

Solutions to Quiz 2 and Problem sheet 2 (non-homework problems)

1 Solutions to Quiz 2

2 Solutions to Problems sheet 2

Problem 1 (HW). Bipartite states, bra-ket notation and partial traces. If we want to write a vector $|v\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ as an element of $\mathbb{C}^{d_1 d_2}$, we order the product standard basis of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ lexicographically. For example, under this ordering $|01\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^4.$$

(a) Let $|\psi_{AB}\rangle = \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^3$ be the vector given by

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

in the lexicographically ordered product basis. Write $|\psi_{AB}\rangle$ in bra-ket notation in the product basis $|ab\rangle$ for $\mathcal{H}_{AB}$.

(b) Let $|\psi_{A'B'}\rangle = \mathcal{H}_{A'B'} = \mathcal{H}_{A'} \otimes \mathcal{H}_{B'} = \mathbb{C}^3 \otimes \mathbb{C}^2$ be the vector given by

$$|\psi_{A'B'}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

in the lexicographically ordered product basis. Write $|\psi_{A'B'}\rangle$ in bra-ket notation in the *computational* for $\mathcal{H}_{A'B'}$.

(c) Let $V_{A'B' \rightarrow AB}$ denote the isometry $\mathcal{H}_{A'B'} \rightarrow \mathcal{H}_{AB}$ that swaps the two subsystems, i.e., $|xy\rangle \mapsto |yx\rangle$ for $x \in \{0, 1, 2\}$ and $y \in \{0, 1\}$. Write down $V_{A'B' \rightarrow AB}|\psi_{A'B'}\rangle \in \mathcal{H}_{AB}$ as a 6-dimensional vector as well as in bra-ket notation.

- (d) Let $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ and $\sigma_{A'B'} = |\psi_{A'B'}\rangle\langle\psi_{A'B'}|$. Compute the reduced density matrices $\rho_A, \rho_B, \sigma_{A'}$ and $\sigma_{B'}$.

Problem 2. Independence and product states. Show that if p_{XY} is a probability distribution, then X and Y are independent if and only if the corresponding density matrix ρ_{XY} is a product state, i.e., $\rho_{XY} = \rho_X \otimes \rho_Y$.

Solution. The density matrix ρ_{XY} corresponding to the joint probability distribution p_{XY} is

$$\rho_{XY} = \sum_{x,y} p_{XY}(x,y) |xy\rangle\langle xy| = \sum_{x,y} p_{XY}(x,y) |x\rangle\langle x| \otimes |y\rangle\langle y|.$$

If X and Y are independent, meaning $p_{XY}(x,y) = p_X(x)p_Y(y)$, then we can factorize:

$$\rho_{XY} = \sum_{x,y} p_X(x) |x\rangle\langle x| \otimes p_Y(y) |y\rangle\langle y| = \left(\sum_x p_X(x) |x\rangle\langle x| \right) \otimes \left(\sum_y p_Y(y) |y\rangle\langle y| \right) = \rho_X \otimes \rho_Y.$$

Conversely, if $\rho_{XY} = \rho_X \otimes \rho_Y$ is a product state, then we have

$$p_{XY}(x,y) = \langle xy | \rho_{XY} | xy \rangle = \langle x | \rho_X | x \rangle \langle y | \rho_Y | y \rangle.$$

So we can define $p_X(x) = \langle x | \rho_X | x \rangle$ and $p_Y(y) = \langle y | \rho_Y | y \rangle$. Finally,

$$\sum_x p_X(x) = \sum_x \langle x | \rho_X | x \rangle = \text{tr}[\rho_X] = 1,$$

and similarly for p_Y , so these are probability distributions. □

Problem 3. Tensor products of operators.

- (a) If $P \in \text{PSD}(\mathcal{H})$ and $Q \in \text{PSD}(\mathcal{K})$, then $P \otimes Q \in \text{PSD}(\mathcal{H} \otimes \mathcal{K})$.
- (b) For any $M \in \text{Lin}(\mathcal{H})$, $N \in \text{Lin}(\mathcal{K})$, we have $\text{tr}[M \otimes N] = \text{tr}[M] \text{tr}[N]$.
- (c) For any $M \in \text{Lin}(\mathcal{H})$, $N \in \text{Lin}(\mathcal{K})$, we have $\text{rank}(M \otimes N) = \text{rank}(M) \text{rank}(N)$.
- (d) If $M \in \text{Lin}(\mathcal{H})$ and $N \in \text{Lin}(\mathcal{K})$ have one of the properties {Hermitian, projection, unitary}, then the tensor product $M \otimes N$ has the property too.

Solution. (a) By the spectral theorem, we have diagonalizations

$$P = \sum_{i=1}^r \lambda_i |\psi_i\rangle\langle\psi_i|, \quad Q = \sum_{j=1}^s \mu_j |\phi_j\rangle\langle\phi_j|.$$

Then

$$P \otimes Q = \sum_{i=1}^r \sum_{j=1}^s \lambda_i \mu_j |\psi_i\rangle\langle\psi_i| \otimes |\phi_j\rangle\langle\phi_j|.$$

We have $\lambda_i \mu_j \geq 0$, for all i, j , since P and Q are positive semidefinite. Also, $\{|\psi_i\rangle \otimes |\phi_j\rangle\}$ forms an orthonormal basis. Thus, $P \otimes Q$ is positive semidefinite on

(b) Let $\{|\psi_i\rangle\}$ and $\{|\phi_j\rangle\}$ be bases of $\mathcal{H}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned}\text{tr}[M \otimes N] &= \sum_{i,j} (\langle \psi_i | \otimes \langle \phi_j |) M \otimes N (|\psi_i\rangle \otimes |\phi_j\rangle) \\ &= \sum_{i,j} \langle \psi_i | M | \psi_i \rangle \langle \phi_j | N | \phi_j \rangle \\ &= \left(\sum_i \langle \psi_i | M | \psi_i \rangle \right) \cdot \left(\sum_j \langle \phi_j | N | \phi_j \rangle \right) \\ &= \text{tr}[M] \cdot \text{tr}[N].\end{aligned}$$

(c) By the singular value decomposition, we have

$$M = \sum_{i=1}^r s_i |e_i\rangle\langle f_i|, \quad N = \sum_{j=1}^s t_j |g_j\rangle\langle h_j|,$$

where $r = \text{rank}(M)$ and $s = \text{rank}(N)$. Further

$$M \otimes N = \sum_i \sum_j s_i t_j |e_i\rangle\langle f_i| \otimes |g_j\rangle\langle h_j| = \sum_{i,j} s_i t_j (|e_i\rangle \otimes |g_j\rangle) (\langle f_i| \otimes \langle h_j|).$$

The latter is a singular value decomposition of $M \otimes N$ and thus $\text{rank}(M \otimes N) = rs$.

(d) We have

$$\begin{aligned}((M \otimes N)(|\psi\rangle \otimes |\phi\rangle))^\dagger &= (M|\psi\rangle \otimes N|\phi\rangle)^\dagger \\ &= (M|\psi\rangle)^\dagger \otimes (N|\phi\rangle)^\dagger \\ &= (\langle \psi | M^\dagger) \otimes (\langle \phi | N^\dagger) \\ &= (\langle \psi | \otimes \langle \phi |) (M^\dagger \otimes N^\dagger).\end{aligned}$$

So, $(M \otimes N)^\dagger = M^\dagger \otimes N^\dagger$, i.e., if M and N are Hermitian, then so is $M \otimes N$.

If $M^2 = M$ and $N^2 = N$, then $(M \otimes N)^2 = M^2 \otimes N^2 = M \otimes N$. So if M and N are projections, then so is $M \otimes N$.

If M and N are unitary, then

$$(M \otimes N)^\dagger (M \otimes N) = (M^\dagger \otimes N^\dagger) (M \otimes N) = M^\dagger M \otimes N^\dagger N = I \otimes I = I,$$

so $M \otimes N$ is also unitary. □

Problem 4 (HW). Product measurement. For measurements $\mu_A: \Omega_1 \rightarrow \text{PSD}(A)$ and $\mu_B: \Omega_2 \rightarrow \text{PSD}(B)$ on quantum systems A and B , the *product measurement* $\mu_A \otimes \mu_B$ is defined by the formula

$$(\mu_A \otimes \mu_B)(x_1, x_2) = \mu_A(x_1) \otimes \mu_B(x_2),$$

for $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$.

In the subproblems (c)–(e) assume that Alice and Bob share a maximally entangled state $|\Phi_{AB}^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

(a) Show that the formula defines a measurement.

- (b) Prove that if we measure any state $\rho_{AB} \in S(AB)$ using the product measurement, then the marginal probability of Alice's outcome $x_1 \in \Omega_1$ is the same as if Bob did not make any measurement at all.
- (c) Suppose that Alice and Bob measure in the standard basis. What is the probability distribution of the outcomes?
- (d) Suppose that Alice and Bob measure in the X -basis $|+\rangle$ and $|-\rangle$. What is the probability distribution of the outcomes?
- (e) In both cases, what does the marginal distribution of the outcomes look like for Alice and Bob? Relate this to their reduced states. Note that while the measurement outcomes are correlated, performing local measurements on a maximally entangled state does not allow them to communicate information.

Problem 5. More properties of partial trace. Let $M_{AB} \in \text{Lin}(AB)$.

- (a) For any $N_B \in \text{Lin}(B)$, we have $\text{tr}_B[(I_A \otimes N_B)M_{AB}] = \text{tr}_B[M_{AB}(I_A \otimes N_B)]$.
- (b) For any unitary $U_B \in \text{Lin}(B)$, we have $\text{tr}_B[(I_A \otimes U_B)M_{AB}(I_A \otimes U_B^\dagger)] = \text{tr}_B[M_{AB}]$.
- (c) For $N_1 \in \text{Lin}(A, C_1)$ and $N_2 \in \text{Lin}(C_2, A)$, we have

$$\text{tr}_B[(N_1 \otimes I_B)M_{AB}(N_2 \otimes I_B)] = N_1 \text{tr}_B[M_{AB}]N_2.$$

- (d) For quantum system A, B , and C , and operator $M_{ABC} \in \text{Lin}(ABC)$, we have

$$\text{tr}_B[\text{tr}_C[M_{ABC}]] = \text{tr}_{BC}[M_{ABC}].$$

Problem 6. Standard purification. We will again prove the lemma about purification in a different way. Suppose that $\rho_A \in S(A)$. Let $|a\rangle$ be a basis for $\mathcal{H}_A$ and let $\mathcal{H}_R = \mathcal{H}_A$. Prove that

$$|\psi_{AR}\rangle = \sum_a (\sqrt{\rho_A} \otimes I_R) |aa\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R$$

is a purification of ρ_A . This is the *standard purification*.

Solution. We just have to check that $\text{tr}_R[|\psi_{AR}\rangle\langle\psi_{AR}|] = \rho_A$. We have

$$\begin{aligned} \text{tr}_R[|\psi_{AR}\rangle\langle\psi_{AR}|] &= \sum_{a,b} \text{tr}_R[(\sqrt{\rho_A} \otimes I_R) |aa\rangle\langle bb| (\sqrt{\rho_A} \otimes I_R)] \\ &= \sum_{a,b} \text{tr}_R[(\sqrt{\rho_A} \otimes I_R) (\sqrt{\rho_A} \otimes I_R) |aa\rangle\langle bb|] \\ &= \sum_{a,b} \text{tr}_R[(\rho_A \otimes I_R) (|a\rangle\langle b| \otimes |a\rangle\langle b|)] \\ &= \sum_{a,b} \text{tr}_R[(\rho_A \otimes I_R) (|a\rangle\langle b| \otimes |a\rangle\langle b|)] \\ &= \sum_{a,b} (\rho_A |a\rangle\langle b|) \text{tr}[|a\rangle\langle b|] \\ &= \sum_a \rho_A |a\rangle\langle a| \\ &= \rho_A. \end{aligned}$$

□

Problem 7. Purity of quantum states. The *purity* of a quantum state ρ is defined as $P(\rho) = \text{tr}[\rho^2]$.

- (a) For $\rho \in S(A)$, $\dim(\mathcal{H}_A) = d$, prove

$$\frac{1}{d} \leq P(\rho) \leq 1.$$

When is equality achieved?

- (b) Let $\rho_{AB} \in S(AB)$ be a pure state with marginal states ρ_A and ρ_B . Show that $P(\rho_A) = P(\rho_B)$.

Solution. (a) Using the spectral decomposition, we can write $\rho = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i|$, where $\{|\psi_i\rangle\}$ is a basis of $\mathcal{H}_A$, $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. Since $\rho^2 = \sum_i \lambda_i^2 |\psi_i\rangle\langle\psi_i|$, we have $\text{tr}[\rho^2] = \sum_i \lambda_i^2 = (\sum_i \lambda_i)^2 - \sum_{i \neq j} \lambda_i \lambda_j = 1 - \sum_{i \neq j} \lambda_i \lambda_j \leq 1$ since $\lambda_i \geq 0$.

Using CS-inequality, we get $(\sum_i \lambda_i)^2 \leq \sum_i \lambda_i^2 \sum_i 1$, which implies

$$\sum_i \lambda_i^2 \geq \frac{1}{d} \left(\sum_i \lambda_i \right)^2 = \frac{1}{d},$$

which gives the lower bound. The equality for the upper bound occurs if and only if ρ is a pure state. The equality for the lower bound occurs if and only if ρ is a maximally mixed state.

- (b) Let $d = \min(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B))$. We apply the Schmidt decomposition and get

$$|\psi_{AB}\rangle = \sum_{i=1}^d s_i |e_i\rangle \otimes |f_i\rangle.$$

The marginal states are given by

$$\rho_A = \sum_{i=1}^d s_i^2 |e_i\rangle\langle e_i|, \quad \rho_B = \sum_{i=1}^d s_i^2 |f_i\rangle\langle f_i|.$$

Therefore, $P(\rho_A) = P(\rho_B)$. □

Problem 8. Marginal problem for maximally entangled states.

- (a) Let $\rho_{AB} \in S(AB)$. Show that if ρ_A is pure, the $\rho_{AB} = \rho_A \otimes \rho_B$. *Hint: consider a purification of ρ_{AB} .*
- (b) Suppose that $\rho_{ABC} \in S(ABC)$ and suppose that ρ_{AB} is pure. Show that $\rho_{BC} = \rho_B \otimes \rho_C$
- (c) Let $\rho_{ABC} \in S(ABC)$ be such that ρ_{AB} and ρ_{BC} are pure. Show that $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$
- (d) Conclude that there can be no state ρ_{ABC} on three qubits such that ρ_{AB} is maximally entangled and ρ_{BC} is maximally entangled.