Solutions to Quiz 1 and Problem sheet 1 (non-homework problems)

1 Solutions to Quiz 1

2 Solutions to Problems sheet 1

Problem 1. Consider the following linear operators.

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \rho_4 = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(a) Which of the ρ_i are density matrices? Which correspond to pure states?
(b) Write ρ_i in bra-ket notation.
(c) Write a spectral decomposition for those ρ_i that are Hermitian.

Solution. (a) All the matrices have trace one, so we need to check PSD. A matrix is PSD if and only if it is Hermitian and has non-negative eigenvalues. Matrices ρ_2, ρ_3 and ρ_4 are Hermitian, so we need to check their eigenvalues. For 2×2 matrices, the eigenvalues sum to the trace and multiply to the determinant. Since the eigenvalues must sum to a positive number (in this case 1) they are non-negative if and only if the determinant is non-negative. Thus, only ρ_2 and ρ_3 are positive semidefinite.

(b)

$$\begin{aligned}
\rho_1 &= |0\rangle\langle 0| + |1\rangle\langle 0|, \\
\rho_2 &= \frac{1}{3}(2|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 1|), \\
\rho_3 &= \frac{1}{2}(|0\rangle\langle 0| - |1\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 1|), \\
\rho_4 &= \frac{1}{2}(|0\rangle\langle 0| + 2|0\rangle\langle 1| + 2|1\rangle\langle 0| + |1\rangle\langle 1|).
\end{aligned}$$

(c)

$$\rho_2 = \lambda_+|\psi_+\rangle\langle\psi_+| + \lambda_-|\psi_-\rangle\langle\psi_-|,$$

where

$$\lambda_\pm = \frac{3 \pm \sqrt{5}}{6}, \quad |\psi_+\rangle = \sqrt{\frac{1+\sqrt{5}}{2\sqrt{5}}}|0\rangle + \sqrt{\frac{2}{(1+\sqrt{5})\sqrt{5}}}|1\rangle,$$

$$|\psi_-\rangle = -\sqrt{\frac{2}{(1+\sqrt{5})\sqrt{5}}}|0\rangle + \sqrt{\frac{1+\sqrt{5}}{2\sqrt{5}}}|1\rangle.$$

$$\rho_3 = |\psi\rangle\langle\psi|,$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$$\rho_4 = \frac{3}{2}|\psi_{3/2}\rangle\langle\psi_{3/2}| - \frac{1}{2}|\psi_{-1/2}\rangle\langle\psi_{-1/2}|,$$

where $|\psi_{3/2}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\psi_{-1/2}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

□

Problem 2. Mixed states as probabilistic mixtures. For each of the following scenarios, write down the density matrix that results from the described procedure. Write down the density matrix both in bra-ket notation and in matrix form. All systems are qubits.

(a) Alice flips a fair coin. If the coin is heads, she prepares the state $|0\rangle$. If the coin lands tails, she perpares $|+\rangle$. You receive the state (but not the result of the coin toss).
(b) Alice measures the state $|0\rangle$ in the X-basis, with outcomes $+$ and $-$. Upon finding oucome $+$, she prepares the state $|0\rangle$, while upon finding $-$, she prepares $|1\rangle$. You receive the state (but not the measurement outcome).
(c) Alice has the state $\frac{1}{\sqrt{5}}(|0\rangle + 2|1\rangle)$ and measures in the standard basis. If she obtains outcome 0, she prepares the state $|0\rangle$. If she obtains outcome $|1\rangle$, she perpares $|+\rangle$ with probability $1/2$ and $|-\rangle$ with probability $1/2$.

Solution. (a) With probability $1/2$, the system is in the state $|0\rangle$, and with probability $1/2$ it is in state $|+\rangle$. So, the density matrix is

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b) By measuring $|0\rangle$ in the X-basis, the probability of getting $+$ is $1/2$ and the probability of getting $-$ is $1/2$. So, the density matrix is

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) We measure $\frac{1}{\sqrt{5}}(|0\rangle + 2|1\rangle)$ in Z-basis. We get

$$\mathbb{P}(0) = \operatorname{tr}[|\psi\rangle\langle\psi| \cdot |0\rangle\langle 0|] = |\langle\psi|0\rangle|^2 = \frac{1}{5}.$$

Similarly, $\mathbb{P}(1) = \frac{4}{5}$. If the first measurement gives 0, she prepares $\rho_0 = |0\rangle\langle 0|$. If the first measurement gives 1, she prepares $\rho_1 = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}I$. Overall, we get

$$\rho = \frac{1}{5}\rho_0 + \frac{4}{5}\rho_1 = \frac{1}{5}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

□