Exercise Sheet 4: Quantum Channels

Select all correct answers and briefly justify them. All Hilbert spaces are finite-dimensional. A quantum channel is a linear, completely positive, trace-preserving map on operators.

Question 1. Which of the following statements are true?

(A) Any map of the form $M \mapsto AMA^\dagger$ is completely positive.
(B) The transpose map $M \mapsto M^T$ is completely positive.
(C) If $\Phi_{A\to B}$ is completely positive, then so is $\Phi_{A\to B} \otimes \mathrm{id}_C$.
(D) The composition of two completely positive maps is completely positive.

Question 2. Which of the following statements are true? Here trace preservation means $\mathrm{tr}[\Phi(M)] = \mathrm{tr}[M]$ for *every* operator M, including non-Hermitian operators.

(A) Any map of the form $M \mapsto AMA^\dagger$ is trace-preserving.
(B) The Hermitian conjugate map $M \mapsto M^\dagger$ is trace-preserving.
(C) The tensor product of two trace-preserving linear maps is trace-preserving.
(D) The composition of two trace-preserving maps is trace-preserving.

Question 3. Which of the following rules on density matrices can be the action of a quantum channel? Assume $d_A = \dim(\mathcal{H}_A) \geq 2$.

(A) $\Phi_{A\to B} : \rho_A \longmapsto \sigma_B \,\mathrm{tr}[\rho_A]$, for a fixed state σ_B on B.
(B) $\Phi_{A\to\mathbb{C}^2} : \rho \longmapsto (|0\rangle\langle 0| \,\mathrm{tr}[\rho^2] + |1\rangle\langle 1|(1 - \mathrm{tr}[\rho^2]))\,\mathrm{tr}[\rho]$.
(C) $\Phi_{\mathbb{C}^2\to\mathbb{C}} : \rho \longmapsto \mathrm{tr}[\rho Z]$, where $Z = \mathrm{diag}(1, -1)$.
(D) $\Phi_{A\to A} : \rho \longmapsto \dfrac{2}{d_A} I_A - \rho$.

Question 4. Consider the replacement channel $\Phi_{A\to B} : M \mapsto \sigma_B \,\mathrm{tr}[M]$ for a fixed state σ_B. Which of the following statements are true? The vectors $\{|a\rangle\}_a$ form an orthonormal basis of A.

(A) $\Phi_{A\to B}$ has a Kraus representation given by $X_a = \sqrt{\sigma_B}\langle a|$.
(B) If $\sigma_B = |\psi\rangle\langle\psi|$ is pure, then $\Phi_{A\to B}$ has a Kraus representation given by $X_a = |\psi\rangle\langle a|$.

Question 5. A machine takes one qubit as its input. With probability 1/2, it applies the Pauli operator X (a bit flip); otherwise, it leaves the state unchanged. Which quantum channel describes the machine? Here I_2 is the qubit identity operator.

(A) $\rho \longmapsto \frac{1}{2} I_2 \operatorname{tr}[\rho]$.
(B) $\rho \longmapsto \frac{1}{2}(\rho + X\rho X)$.
(C) $\rho \longmapsto \frac{1}{2}\left(\rho + \frac{1}{2} I_2 \operatorname{tr}[\rho]\right)$.
(D) $\rho \longmapsto \frac{1}{2}\left(X\rho X + \frac{1}{2} I_2 \operatorname{tr}[\rho]\right)$.

Question 6. Let Φ be the channel from Question 5. What is its action on the Bloch sphere?

(A) Reflection in the yz-plane.
(B) Projection onto the yz-plane.
(C) Rotation by π radians about the x-axis.
(D) Projection onto the x-axis.

Question 7. A machine takes one qubit as its input and measures it in the Z basis $\{|0\rangle, |1\rangle\}$. If the outcome is 0, it prepares the state $|\psi\rangle$; otherwise, it prepares $|\phi\rangle$. Assume both vectors are normalized. Which quantum channel describes the machine?

(A) $\rho \longmapsto \langle 0|\rho|0\rangle |\psi\rangle\langle\psi| + \langle 1|\rho|1\rangle |\phi\rangle\langle\phi|$.
(B) $\rho \longmapsto \langle \psi|\rho|\psi\rangle |0\rangle\langle 0| + \langle \phi|\rho|\phi\rangle |1\rangle\langle 1|$.
(C) $\rho \longmapsto \frac{1}{2}(\langle 0|\rho|0\rangle |\psi\rangle\langle\psi| + \langle 1|\rho|1\rangle |\phi\rangle\langle\phi|)$.
(D) $\rho \longmapsto \frac{1}{2}(\langle \psi|\rho|\psi\rangle |0\rangle\langle 0| + \langle \phi|\rho|\phi\rangle |1\rangle\langle 1|)$.

Solutions

1. **A, C, and D.** For any positive W on a reference and the input,
$$(\mathrm{id}_R \otimes \Phi)(W) = (I_R \otimes A)W(I_R \otimes A)^\dagger \geq 0$$
when $\Phi(M) = AMA^\dagger$. Tensoring a completely positive map with an identity preserves complete positivity, as does composition. Transposition fails the test on half of a Bell state: the partial transpose has an eigenvalue $-1/2$.

2. **C and D.** A single-Kraus map preserves trace exactly when $A^\dagger A = I$, so A is false without that condition. Also, $\mathrm{tr}[M^\dagger] = \overline{\mathrm{tr}[M]}$, which need not equal $\mathrm{tr}[M]$: take $M = iI$. The adjoint map is not complex-linear either. Trace preservation of a tensor product follows first on product operators from $\mathrm{tr}[M \otimes N] = \mathrm{tr}[M]\,\mathrm{tr}[N]$, then on all operators by linearity. Trace preservation is also preserved under composition.

3. **A only.** The replacement map is a channel, for example by the Kraus construction below. B does not respect mixtures: two orthogonal pure inputs each give $|0\rangle\langle 0|$, but their equal mixture has purity $1/2$ and gives $I_2/2$. C sends $|1\rangle\langle 1|$ to -1, so it is not positive. For D, a pure input gives an eigenvalue $2/d_A - 1 < 0$ if $d_A > 2$. If $d_A = 2$, its linear extension is $M \mapsto \mathrm{tr}[M]I_2 - M$. Its Choi operator is
$$I_2 \otimes I_2 - |\Omega\rangle\langle\Omega|, \qquad |\Omega\rangle = |00\rangle + |11\rangle,$$
which has eigenvalue -1 on $|\Omega\rangle$ and is not positive.

4. **B only.** The expression in A is generally not a well-defined operator from A to B: $\sqrt{\sigma_B}$ is an operator on B, while $\langle a|$ maps A to $\mathbb{C}$. If $\sigma_B = |\psi\rangle\langle\psi|$, then
$$\sum_a X_a M X_a^\dagger = \sum_a \langle a|M|a\rangle |\psi\rangle\langle\psi| = \sigma_B\,\mathrm{tr}[M], \qquad \sum_a X_a^\dagger X_a = I_A.$$
For general $\sigma_B = \sum_b s_b |b_B\rangle\langle b_B|$, a correct Kraus family is $X_{ba} = \sqrt{s_b}|b_B\rangle\langle a|$.

5. **B.** Averaging the two possible unitary evolutions gives $\Phi(\rho) = \frac{1}{2}\rho + \frac{1}{2}X\rho X$. Its Kraus operators are $I_2/\sqrt{2}$ and $X/\sqrt{2}$.

6. **D.** For $\rho = \frac{1}{2}(I_2 + r_x X + r_y Y + r_z Z)$, conjugation by X sends (r_x, r_y, r_z) to $(r_x, -r_y, -r_z)$. Averaging the original and conjugated state gives $(r_x, 0, 0)$.

7. **A.** The Born probabilities are $\langle 0|\rho|0\rangle$ and $\langle 1|\rho|1\rangle$. Weighting the two prepared states by these probabilities gives A. Equivalently, use Kraus operators $K_0 = |\psi\rangle\langle 0|$ and $K_1 = |\phi\rangle\langle 1|$; their products satisfy $K_0^\dagger K_0 + K_1^\dagger K_1 = I_2$. These answers describe the prescribed machine for arbitrary normalized $|\psi\rangle, |\phi\rangle$; special choices can make some displayed formulas coincide.