Exercise Sheet 3: Non-local Games

Recall that in the CHSH game Alice and Bob receive independently and uniformly random questions $x, y \in \{0, 1\}$, respectively, and produce answers $a, b \in \{0, 1\}$. They win if

$$xy = a + b \pmod 2.$$

They cannot communicate after receiving their questions. Select all correct answers and briefly justify them.

Question 1. Which of the following statements about the CHSH game are true?

(A) Randomized strategies can achieve a higher winning probability than deterministic strategies.
(B) Quantum strategies can achieve a higher winning probability than classical randomized strategies.

Question 2. Suppose that Alice and Bob share a bipartite state ρ_{AB}. Upon receiving their questions, they perform local measurements μ_A^x and μ_B^y, respectively. Which of the following statements are true?

(A) The order in which Alice and Bob perform their measurements does not affect their joint outcome probabilities.
(B) For fixed questions, Alice's and Bob's measurement outcomes are independent whenever ρ_{AB} is not entangled.

Question 3. Suppose that Alice and Bob share a quantum state of the form

$$\rho_{AB} = \sum_{\lambda \in \Lambda} p_\lambda \rho_{A,\lambda} \otimes \rho_{B,\lambda},$$

where $\rho_{A,\lambda}$ and $\rho_{B,\lambda}$ are quantum states and $\{p_\lambda\}_{\lambda \in \Lambda}$ is a probability distribution. What is their optimal winning probability in the CHSH game?

(A) $\frac{1}{2}$.
(B) $\frac{3}{4}$.
(C) $\frac{1}{2} + \frac{1}{2\sqrt{2}}$.
(D) 1.

Question 4. Suppose that Alice and Bob share the pure quantum state

$$|\psi_{AB}\rangle = \frac{1}{2}|0\rangle \otimes (|0\rangle + |1\rangle) + \frac{1}{2}|1\rangle \otimes (|0\rangle - |1\rangle).$$

What is their optimal winning probability in the CHSH game?

(A) $\frac{1}{2}$.
(B) $\frac{3}{4}$.
(C) $\frac{1}{2} + \frac{1}{2\sqrt{2}}$.
(D) 1.

Solutions

1. **B only.** A randomized strategy is a mixture of deterministic strategies, so its winning probability cannot exceed the best deterministic value. For CHSH the classical value is $3/4$, whereas the quantum value is $1/2 + 1/(2\sqrt{2}) > 3/4$.

2. **A only.** Local measurements act on different tensor factors. Their joint probabilities are
$$p(a, b \mid x, y) = \mathrm{tr}[(\mu_A^x(a) \otimes \mu_B^y(b))\rho_{AB}],$$
independently of their order. Separability does not imply independence: the separable state $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ gives perfectly correlated computational-basis outcomes. In particular, $p(0,0) = 1/2 \neq p_A(0)p_B(0) = 1/4$.

3. **B:** $3/4$**.** The measurement statistics of the separable state have the form
$$p(a, b \mid x, y) = \sum_\lambda p_\lambda \,\mathrm{tr}[\mu_A^x(a)\rho_{A,\lambda}]\,\mathrm{tr}[\mu_B^y(b)\rho_{B,\lambda}].$$
This is a classical strategy with shared randomness λ, so its winning probability is at most $3/4$. Alice and Bob can attain $3/4$ by ignoring the state and always answering $a = b = 0$.

4. **C:** $1/2 + 1/(2\sqrt{2})$**.** With $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, the state is
$$|\psi_{AB}\rangle = \frac{|0\rangle|+\rangle + |1\rangle|-\rangle}{\sqrt{2}} = (I \otimes H)|\Phi^+\rangle, \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$
Bob can apply H to recover the Bell state $|\Phi^+\rangle$ and then use the optimal CHSH strategy. Tsirelson's bound shows that no larger winning probability is possible.