Problem sheet 6: Distance Measures

Problem 1. Distances between states. Find the trace distance and the fidelity between the following single-qubit states.

(a) $\rho = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$ and $\sigma = |0\rangle\langle 0|$.

(b) $\rho = \frac{1}{3}|+\rangle\langle +| + \frac{2}{3}|-\rangle\langle -|$ and $\sigma = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$.

(c) $\rho = \frac{1}{11}(5|0\rangle\langle 0| + 6|1\rangle\langle 1| - 4|0\rangle\langle 1| - 4|1\rangle\langle 0|)$ and $\sigma = \frac{1}{3}(|0\rangle\langle 0| + 2|1\rangle\langle 1| + |1\rangle\langle 0| + |0\rangle\langle 1|)$.

Hint:

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^2 = 5\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Problem 2 (HW)**. Helstrom's theorem.** Let $\rho, \sigma \in S(\mathcal{H})$. Suppose that with probability 1/2 we received the state ρ and with probablity 1/2 we received the state σ. Prove that the optimlal probability of indentifying the correct state by a two-outcome measurement is given by

$$p_{\text{opt}} = \frac{1}{2} + \frac{1}{2}T(\rho, \sigma).$$

Hint: Take any two-outcome measurement μ, where outcome 0 means that we guess that the state is ρ and outcome 1 means that we guess the state σ. Find the probability of guessing correctly; this should be a formula depending on μ, ρ and σ. Use the variational characterization of trace distance to get the theorem.

Problem 3. Properties of the fidelity. Suppose $\rho, \sigma \in S(\mathcal{H})$. Prove the following properties of the fidelity.

(a) $0 \leq F(\rho, \sigma) \leq 1$ and $F(\rho, \sigma) + 1$ if and only if $\rho = 0$.

(b) The fidelity is invariant under isometries: if $V \in \text{Isom}(\mathcal{H}, \mathcal{K})$, then

$$F(V\rho V^\dagger, V\sigma V^\dagger) = F(\rho, \sigma).$$

(c) The fidelity is monotonic under quantum channels: if $\Phi_{A\to B} \in C(A, B)$ and $\rho_A, \sigma_A \in S(A)$, then

$$F(\Phi_{A\to B}[\rho_A], \Phi_{A\to B}[\sigma_A]) \geq F(\rho_A, \sigma_A).$$

Problem 4. Fuchs-van de Graaf inequalities.

(a) Give $\rho_A, \sigma_A \in S(A)$, argue that there eixst purifications ρ_{AR}, σ_{AR} of ρ_A and σ_A such that

$$T(\rho_{AR}, \sigma_{AR}) = \sqrt{1 - F(\rho_A, \sigma_A)^2}$$

and use this to show that

$$T(\rho_A, \rho_A) \leq \sqrt{1 - F(\rho_A, \sigma_A)^2}.$$

(b) Show that for any probability distritubtions p_X and q_X, we have

$$1 - F(p_X, q_X) \leq T(p_X, q_X).$$

(c) Show that

$$1 - F(\rho_A, \sigma_A) \leq T(\rho_A, \sigma_A)$$

where you can use the fact that there exists some measurment such that if p_X and q_X denote the outcome probabilities after mesuring ρ_A and σ_A, it holds that $F(\rho_A, \sigma_A) = F(p_X, q_X)$.

Problem 5. Optimally distinguishing between quantum states. Let $\rho \in S(A)$ be a pure state, $\rho = |\psi\rangle\langle\psi|$ and let $\tau = \frac{1}{\dim \mathcal{H}_A} \mathbb{1}_A$ be the maximally mixed state on A.

(a) Show that ρ can be distinguished from τ using a two-outcome measurment with optimal probability

$$p_{\text{opt}} = \frac{2 \dim \mathcal{H}_a - 1}{2 \dim \mathcal{H}_A}.$$

(b) Write down the measurement that optimally distinguishes ρ from τ in this case.

(c) Let σ be a general state. Using a similar measurment, show that ρ and σ can be distinguished by a two-outcome measurment can be distinguished by a two-outcome measurment with probability $1 - \frac{1}{2}\langle\psi|\sigma|\psi\rangle$.

Deduce that in the case when one of our states is pure we can obtain the following improvement on the Fuchs-vande Graaf lower bound:

$$1 - F(\rho, \sigma)^2 \leq T(\rho, \sigma).$$