Problem sheet 5: Basic Quantum Information Protocols

Problem 1 (HW)**. Entanglement swapping.** Suppose that Alice and Bob both share a maximally entangled state with Charlie. Describe the procedure by which Alice and Bob can generate a maximally entangled qubit pair by only acting on their local quantum system and exchanging classical bits.

Problem 2. Error correction and Kraus operators.

(a) Suppose that an error correcting code V can correct errors from a noise channel with Kraus operators $\{X_i\}$. Show that it can also correct errors for any channel which has Kraus operators $\{Y_j\}$, where each Y_j is a linear combination of X_i's.

(b) Consider an error correcting code on a system B of n qubits, with P the projection operator on the code subspace. If M is a qubit operator, let M_i denote the operator which acts as M on the i-th qubit and as $\mathbb{1}$ on all other. Suppose that PM_iN_jP is proportional to P, for all $i, j = 1, \ldots, n$ and $M, N \in \{I, X, Y, Z\}$ are arbitrary Pauli operators. Show that we can correct an arbitrary error channel of single-qubit errors, i.e., any channel of the form

$$\Phi_B = \sum_{i=1}^{n} p_i \Phi_i,$$

where $\{p_i\}$ is a probability distribution and Φ_i is a channel which acts only on the i-th qubit.

Problem 3. Remote state preparation. This is about a protocol callled *remote state preparation*, which is closely related to quantum teleportation, but here only one bit of classical communication is required to remotely prepare a given qubit state. Compared to the teleportation, the sender knows a classicla description of the state to prepare, and has access to a larger number of entangled qubits.

(a) Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$ be a pure qubit state. Show that

$$(|\psi\rangle\langle\psi|)^T = |\overline{\psi}\rangle\langle\overline{\psi}|, \quad \text{and} \quad \mathbb{1} - |\overline{\psi}\rangle\langle\overline{\psi}| = |\overline{\psi}^\perp\rangle\langle\overline{\psi}^\perp|,$$

where the transpose is with respect to the computational basis, and $|\overline{\psi}\rangle = \overline{\alpha}|0\rangle + \overline{\beta}|1\rangle$, $|\overline{\psi}^\perp\rangle = \overline{\beta}|0\rangle - \overline{\alpha}|1\rangle$.

(b) Suppose that Alice and Bob share a maximally entangled state $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice would like to give Bob the state $|\psi\rangle$, but she does not want to give Bob any information about α and β.

Alice performs a projective measurment on her part of the maximally entangled state, corresponding to the projectors $\Pi_0 = |\overline{\psi}\rangle\langle\overline{\psi}|$ and $\Pi_1 = \mathbb{1} - \Pi_0$. Show that the outcome probabilities of this measurement are both $1/2$.

(c) Alice then sends Bob the single-bit outcome of her measurment x in a classical system C. Show that Bob now holds the state

$$\rho_{BC} = \frac{1}{2}|\psi_B\rangle\langle\psi_B| \otimes |0_C\rangle\langle 0_C| + \frac{1}{2}|\psi_B^\perp\rangle\langle\psi_B^\perp| \otimes |1_C\rangle\langle 1_C|,$$

where $|\psi^\perp\rangle = \beta|0\rangle - \alpha|1\rangle$.

(d) Assumme, for the moment, that $|\psi\rangle$ is of the form $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$. Show that

$$Z|\psi^\perp\rangle\langle\psi^\perp|Z = |\psi\rangle\langle\psi|,$$

and hence describe how Bob can recover the state $|\psi\rangle$ from ρ_{BC}.

(e) Suppose that Alice wants to send Bob many qubits $|\psi_1\rangle, \ldots, |\psi_n\rangle \in \mathbb{C}^2$, which do not necessarilty take the above form. Assume that Alice and Bob share $n \cdot m$ maximally entangled states, where $m = 2^{n+\log n}$. They each arrange their qubits in a rectangle, so that the qubit from the (i,j)-th maximally entangled pair lies in the i-th row and j-th column. For each i, $i = 1, \ldots, n$, Alice measures the entire i-th row of qubits in the $\left\{|\overline{\psi_i}\rangle, |\overline{\psi_i}^\perp\rangle\right\}$ basis. Show that, with high probability, there will be an entire column of qubits for which the measurements were succesful, that is, the result corresponded to the projector $\Pi_0 = |\overline{\psi}\rangle\langle\overline{\psi}|$.

(f) Alice sends Bob the index $j = 1, \ldots, m$ of such column classically. Show that in this way, Alice can remotely prepare n states in Bob's system with approximately 1 bit of classical communication per state, in the limit as $n \to \infty$.