Problem sheet 2: Multiple Quantum Systems

Problem 1 (HW). **Bipartite states, bra-ket notation and partial traces.** If we want to write a vector $|v\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ as an element of $\mathbb{C}^{d_1 d_2}$, we order the product standard basis of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ lexicographically. For example, under this ordering $|01\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^4.$$

(a) Let $|\psi_{AB}\rangle = \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^3$ be the vector given by

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

in the lexicographically ordered product basis. Write $|\psi_{AB}\rangle$ in bra-ket notation in the product basis $|ab\rangle$ for $\mathcal{H}_{AB}$.

(b) Let $|\psi_{A'B'}\rangle = \mathcal{H}_{A'B'} = \mathcal{H}_{A'} \otimes \mathcal{H}_{B'} = \mathbb{C}^3 \otimes \mathbb{C}^2$ be the vector given by

$$|\psi_{A'B'}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

in the lexicographically ordered product basis. Write $|\psi_{A'B'}\rangle$ in bra-ket notation in the *computational* for $\mathcal{H}_{A'B'}$.

(c) Let $V_{A'B' \to AB}$ denote the isometry $\mathcal{H}_{A'B'} \to \mathcal{H}_{AB}$ that swaps the two subsystems, i.e., $|xy\rangle \mapsto |yx\rangle$ for $x \in \{0, 1, 2\}$ and $y \in |0, 1\rangle$. Write down $V_{A'B' \to AB} |\psi_{A'B'}\rangle \in \mathcal{H}_{AB}$ as a 6-dimensional vector as well as in bra-ket notation.

(d) Let $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ and $\sigma_{A'B'} = |\psi_{A'B'}\rangle\langle\psi_{A'B'}|$. Compute the reduced density matrices ρ_A, ρ_B, $\sigma_{A'}$ and $\sigma_{B'}$.

Problem 2. Independence and product states. Show that if p_{XY} is a probability distribution, then X and Y are independent if and only if the corresponding density matrix ρ_{XY} is a product state, i.e., $\rho_{XY} = \rho_X \otimes \rho_Y$.

Problem 3. Tensor products of operators.

(a) If $P \in \mathrm{PSD}(\mathcal{H})$ and $Q \in \mathrm{PSD}(\mathcal{K})$, then $P \otimes Q \in \mathrm{PSD}(\mathcal{H} \otimes \mathcal{K})$.

(b) For any $M \in \mathrm{Lin}(\mathcal{H})$, $N \in \mathrm{Lin}(\mathcal{K})$, we have $\mathrm{tr}[M \otimes N] = \mathrm{tr}[M]\,\mathrm{tr}[N]$.

(c) For any $M \in \mathrm{Lin}(\mathcal{H})$, $N \in \mathrm{Lin}(\mathcal{K})$, we have $\mathrm{rank}(M \otimes N) = \mathrm{rank}(M)\,\mathrm{rank}(N)$.

(d) If $M \in \mathrm{Lin}(\mathcal{H})$ and $N \in \mathrm{Lin}(\mathcal{K})$ have one of the properties {Hermitian, projection, unitary}, then the tensor product $M \otimes N$ has the property too.

Problem 4 (HW)**. Product measurement.** For measurements $\mu_A \colon \Omega_1 \to \mathrm{PSD}(A)$ and $\mu_B \colon \Omega_2 \to \mathrm{PSD}(B)$ on quantum systems A and B, the *product measurement* $\mu_A \otimes \mu_B$ is defined by the formula

$$(\mu_A \otimes \mu_B)(x_1, x_2) = \mu_A(x_1) \otimes \mu_B(x_2),$$

for $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$.

In the subproblems (c)–(e) assume that Alice and Bob share a maximally entangled state $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

(a) Show that the formula defines a measurement.

(b) Prove that if we measure any state $\rho_{AB} \in S(AB)$ using the product measurement, then the marginal probability of Alice's outcome $x_1 \in \Omega_1$ is the same as if Bob did not make any measurement at all.

(c) Suppose that Alice and Bob measure in the standard basis. What is the probability distribution of the outcomes?

(d) Suppose that Alice and Bob measure in the X-basis $|+\rangle$ and $|-\rangle$. What is the probability distribution of the outcomes?

(e) In both cases, what does the marginal distribution of the outcomes look like for Alice and Bob? Relate this to their reduced states. Note that while the measurement outcomes are correlated, performing local measurements on a maximally engangled state does not allow them to communicate information.

Problem 5. More properties of partial trace. Let $M_{AB} \in \mathrm{Lin}(AB)$.

(a) For any $N_B \in \mathrm{Lin}(B)$, we have $\mathrm{tr}_B[(I_A \otimes N_B)M_{AB}] = \mathrm{tr}_B[M_{AB}(I_A \otimes N_B)]$.

(b) For any unitary $U_B \in \mathrm{Lin}(B)$, we have $\mathrm{tr}_B[(I_A \otimes U_B)M_{AB}(I_A \otimes U_B^\dagger)] = \mathrm{tr}_B[M_{AB}]$.

(c) For $N_1 \in \mathrm{Lin}(A, C_1)$ and $N_2 \in \mathrm{Lin}(C_2, A)$, we have

$$\mathrm{tr}_B[(N_1 \otimes I_B)M_{AB}(N_2 \otimes I_B)] = N_1\,\mathrm{tr}_B[M_{AB}]N_2.$$

(d) For quantum system A, B, and C, and operator $M_{ABC} \in \mathrm{Lin}(ABC)$, we have

$$\mathrm{tr}_B[\mathrm{tr}_C[M_{ABC}]] = \mathrm{tr}_{BC}[M_{ABC}].$$

Problem 6. Standard purification. We will again prove the lemma about purification in a different way. Suppose that $\rho_A \in S(A)$. Let $|a\rangle$ be a basis for $\mathcal{H}_A$ and let $\mathcal{H}_R = \mathcal{H}_A$. Prove that

$$|\psi_{AR}\rangle = \sum_a (\sqrt{\rho_A} \otimes I_R)|aa\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R$$

is a purification of ρ_A. This is the *standard purification.*

Problem 7. Purity of quantum states. The *purity* of a quantum state ρ is defined as $P(\rho) = \mathrm{tr}[\rho^2]$.

(a) For $\rho \in S(A)$, $\dim(\mathcal{H}_A) = d$, prove

$$\frac{1}{d} \leq P(\rho) \leq 1.$$

When is equality achived?

(b) Let $\rho_{AB} \in S(AB)$ be a pure state with marginal states ρ_A and ρ_B. Show that $P(\rho_A) = P(\rho_B)$.

Problem 8. Marginal problem for maximally enganlged states.

(a) Let $\rho_{AB} \in S(AB)$. Show that if ρ_A is pure, the $\rho_{AB} = \rho_A \otimes \rho_B$. *Hint: consider a purification of* ρ_{AB}.

(b) Suppose that $\rho_{ABC} \in S(ABC)$ and suppose that ρ_{AB} is pure. Show taht $\rho_{BC} = \rho_B \otimes \rho_C$

(c) Let $\rho_{ABC} \in S(ABC)$ be such that ρ_{AB} are ρ_{BC} are pure. Show that $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$

(d) Conclude that there can be no state ρ_{ABC} on three qubits such that ρ_{AB} is maximally entangled and ρ_{BC} is maximally entangled.