Problem sheet 1: Quantum States

Problem 1. Consider the following linear operators.

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \rho_4 = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(a) Which of the ρ_i are density matrices? Which correspond to pure states?
(b) Write ρ_i in bra-ket notation.
(c) Write a spectral decomposition for those ρ_i that are Hermitian.

Problem 2. Mixed states as probabilistic mixtures. For each of the following scenarios, write down the density matrix that results from the described procedure. Write down the density matrix both in bra-ket notation and in matrix form. All systems are qubits.

(a) Alice flips a fair coin. If the coin is heads, she prepares the state $|0\rangle$. If the coin lands tails, she perpares $|+\rangle$. You receive the state (but not the result of the coin toss).
(b) Alice measures the state $|0\rangle$ in the X-basis, with outcomes $+$ and $-$. Upon finding oucome $+$, she prepares the state $|0\rangle$, while upon finding $-$, she prepares $|1\rangle$. You receive the state (but not the measurement outcome).
(c) Alice has the state $\frac{1}{\sqrt{5}}(|0\rangle + 2|1\rangle)$ and measures in the standard basis. If she obtains outcome 0, she prepares the state $|0\rangle$. If she obtains outcome $|1\rangle$, she perpares $|+\rangle$ with probability $1/2$ and $|-\rangle$ with probability $1/2$.

Problem 3 (HW)**. Convexity of $S(\mathcal{H})$.** The goal is to prove that the set of density matrices $S(\mathcal{H})$ is convex and that the set of extreme points coincides with the set of pure states. Possible approach:

(a) Prove that $S(\mathcal{H})$ is a convex subset of $\mathrm{Lin}(\mathcal{H})$.
(b) Show that any state which is *not* pure is not extremal.
(c) Show that every pure state is extremal. *Hint: suppose that a pure state can be written as a convex combination, then use CS-inequality.*

Problem 4 (HW)**. Measurements and the Bloch sphere.** This question is about measurements of qubit states.

(a) If $r = (x, y, z)$ with $\|r\| = 1$, show that the pure quantum states corresponding to r and $-r$ are orthogonal, and that they are eigenvectors with eigenvalues ± 1 of the operator

$$xX + yY + zZ.$$

(b) Conclude that μ_r as defined in Section 1.5 of the lecture notes defines a two-outcome measurement.

(c) Consider a state parametrized by $s = (x', y', z')$ in the Bloch ball. Show that the probability of obtaining outcome 0 from the measurement μ_r is given by

$$p(0) = \frac{1}{2} + \frac{1}{2} r \cdot s = \frac{1}{2} + \frac{1}{2}(xx' + yy' + zz').$$

(d) Consider the following set-up: You have access to a source producing an unknown state ρ, and you can do arbitrary qubit measurements. You can repreat many times, with different measurements settings; each time you receive the same state ρ. You would like to learn the state ρ, that is, you would like to learn $s = (x', y', z')$ such that $\rho \approx \rho(s)$. Suppose you measure N times along z-axis (so using $r = (0, 0, 1)$), obtaining N_0 times outcome 0 and N_1 times outcome 1. What values do you expect for N_0/N and N_1/N for large N?

(e) Argue that a reasonable estimate for z' is given by $(N_0 - N_1)/N$.

(f) Describe a procedure to estimate the unknown state ρ.