Homework sheet 3: Non-local Games

Problem 1. The GHZ game. Let $\mathcal{H}_{ABC} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ be the Hilbert space of three qubits shared by Alice, Bob, and Charlie. The *Greenberger–Horne–Zeilinger (GHZ) state* is

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

(a) Show that the reduced state on A is maximally mixed:

$$\mathrm{tr}_{BC}[|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|] = \frac{1}{2} I_A.$$

(b) Define

$$M_0 = X \otimes X \otimes X, \qquad M_1 = X \otimes Y \otimes Y,$$
$$M_2 = Y \otimes X \otimes Y, \qquad M_3 = Y \otimes Y \otimes X,$$

where X and Y are the Pauli matrices. Show that $|\mathrm{GHZ}\rangle$ is an eigenvector of each M_i, and determine the corresponding eigenvalues λ_i.

The GHZ game is a three-player non-local game. The referee chooses a question triple uniformly from

$$\mathcal{Q} = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$$

and sends x to Alice, y to Bob, and z to Charlie. Without communicating, they output bits a, b, c, respectively. They win if

$$x \vee y \vee z = a \oplus b \oplus c,$$

where $\vee$ is logical OR and $\oplus$ is addition modulo 2. Thus the required output parity is 0 for $(0,0,0)$ and 1 for the other three question triples.

(c) Suppose the players use a classical deterministic strategy $a = f(x)$, $b = g(y)$, $c = h(z)$. By considering

$$\sum_{(x,y,z)\in\mathcal{Q}} \big(f(x) + g(y) + h(z)\big) \pmod 2,$$

show that the optimal classical winning probability is $\omega_c(\mathrm{GHZ}) = 3/4$. Explain why shared randomness cannot improve this value.

(d) Now suppose the players share $|\mathrm{GHZ}\rangle$. Each player measures X on question 0 and Y on question 1. They answer 0 for measurement outcome $+1$ and 1 for outcome -1, so the three measured eigenvalues are $(-1)^a$, $(-1)^b$, and $(-1)^c$. Show that this strategy wins with certainty, and hence $\omega_q(\mathrm{GHZ}) = 1$.