

GAMES AND ALGEBRAS

SAMUEL J. HARRIS AND VERN I. PAULSEN

ABSTRACT. We present an overview of the theory of nonlocal games and how games induce algebras. These algebras have been used to separate various sets of quantum correlations, leading to the resolution of problems of Connes, Kirchberg and Tsirelson. We survey the theory of various families of games, including games arising from graph isomorphisms, graph colorings, and systems of equations.

CONTENTS

1. Introduction	2
2. Two Person Cooperative Games	3
2.1. The CHSH Game	4
2.2. Random Strategies, Perfect Strategies, and the Classical Value of a Game	5
2.3. Local Densities	6
2.4. Multiplayer Games and Proof Verification	8
3. Models for Quantum Correlations: Tsirelson's Problems	8
3.1. Nonsignalling Densities	11
3.2. Quantum Values of Games	12
3.3. POVM's versus PVM's	12
3.4. Densities and States on Tensor Products of Free Group Algebras	13
3.5. The Problems of Connes and Kirchberg	16
3.6. Algebraic Conditions for Perfect qc-Strategies	19
3.7. Multipartite Correlations and Real Versus Complex Models	19
4. Synchronous Games and Synchronous Correlations	20
4.1. The Fundamental Orthogonality Relations of a Synchronous Game	22
4.2. The Algebra of a Synchronous Game	23
4.3. Synchronous Values of Games	25
4.4. Multipartite Theory of Synchronous Densities	26
4.5. Beyond Synchronous	27
5. XOR Games	27
6. Linear Constraint System Games	28
6.1. Linear equations over $\mathbb{Z}$	30

Key words and phrases. quantum correlation, nonlocal game, Connes' embedding problem, trace, synchronous, chromatic number, graph isomorphism .

6.2. Other Systems of Equations	30
7. Games Based on Graphs	31
7.1. The Graph Homomorphism Game	31
7.2. Universality of the 3-coloring Game	33
7.3. Hadamard Graphs	33
7.4. Fractional Chromatic Numbers	34
7.5. The Algebra of the Graph Coloring Game	36
7.6. Locally Commuting Algebras for Graphs	36
7.7. Multiparty Graph Coloring	36
8. The Graph Isomorphism Game	36
8.1. The Quantum Permutation Group	38
8.2. Quantum Analogues of Doubly Stochastic Matrices	39
8.3. Bisynchronous Games	39
9. Operations on Games	40
10. Games with Quantum Questions and Answers	42
10.1. Quantum XOR games	44
10.2. Quantum games for quantum graphs	46
References	48

1. INTRODUCTION

Historically, groups and group like objects, such as groupoids and dynamical systems, have had various algebras affiliated with them and these correspondences have led to a beautiful area often referred to as *noncommutative harmonic analysis*.

Recently, certain types of games have had algebras affiliated with them and this correspondence has been used to construct algebras with elusive properties. The most famous example is the recent solution [JNV⁺20] of the *Connes' Embedding Problem* by constructing a game whose affiliated tracial C^* -algebra is not embeddable in the sense proposed by Connes [Con76]. The construction of such an algebra had eluded operator algebraists since the 1980's. So the construction of algebras from games is turning into a powerful tool and the mathematics surrounding this area is likely to continue to develop.

The idea of using the type of games that we will be studying as a means of checking proofs has been an active area of computer science since at least the 1980's [BGKW88].

The purpose of these notes is to provide an introduction to this area, together with a bit of the historical motivations. They are adapted from the second author's 2021 Copenhagen Masterclass.

For the purposes of clarity and in an attempt to unify results, we often adopt notations that are different from the literature.

We begin with an exposition of the mathematics surrounding this area, especially the relationship with the Connes' Embedding Problem, and then

focus on various families of games and what is known about them. Readers impatient to see some games might want to jump around a bit.

There is already a tremendous literature in this area and we apologize in advance for our many omissions. Generally, we have tended to cite more recent works, which contain citations to the earlier works. Hopefully, we have included enough leads that a diligent scholar will be able to uncover these other treasures.

We would also like to thank the anonymous referees and Ken Dykema for many helpful comments that improved our exposition.

2. TWO PERSON COOPERATIVE GAMES

The types of games that we shall be interested in are *two person games*, which are *cooperative*, *memoryless* and the players are *non-communicating*. Generally, the two players are referred to as Alice and Bob. Intuitively, in such a game the two players are playing cooperatively to give correct pairs of answers to pairs of questions posed by a third party often called the *Referee* or *Verifier*. Whether the pair of answers returned by the players is satisfactory or not depends not just on the individual answers but on the 4-tuple consisting of the question-answer pair.

Such a game is described by two *input* sets I_A, I_B , often thought of as sets of questions, two *output* sets O_A, O_B , often thought of as answers, and a function

$$V : O_A \times O_B \times I_A \times I_B \rightarrow \{0, 1\},$$

often called the *rules* or *verification function*, where

$$W := \{(a, b, x, y) : V(a, b, x, y) = 1\},$$

is the set of *correct* or *winning* 4-tuples and

$$L := \{(a, b, x, y) : V(a, b, x, y) = 0\},$$

is the set of *incorrect* or *losing* 4-tuples.

For each *round* of the game Alice and Bob receive an input pair (x, y) and return an output pair (a, b) .

When we say that the players are not allowed to *communicate* or that the game is *non-communicating*, this means that Alice must return her answer without knowing the question y that Bob was asked and without knowing the answer b that Bob gave. Similarly, Bob does not know Alice's question-answer pair.

When we say that the game is *memoryless*, we mean that if Alice and Bob receive the same input pair (x, y) at two different rounds of the game, then there is no penalty if they return different pairs of outputs for the different rounds.

Thus, a game $\mathcal{G}$ is specified by (I_A, I_B, O_A, O_B, V) . Before the game begins Alice and Bob know the four sets and the function V . Even though Alice and Bob are not allowed to communicate during the game they are allowed to communicate before the game and decide on some type of strategy.

When we want to talk about the probability of winning such a game we also need to specify a probability density, i.e., a prior distribution, on input pairs, i.e., a function $\pi : I_A \times I_B \rightarrow [0, 1]$ such that

$$\sum_{x \in I_A, y \in I_B} \pi(x, y) = 1.$$

For games with densities, Alice and Bob also know the density before the start of the game. We denote a game with density by $(\mathcal{G}, \pi)$.

A **deterministic strategy** for such a game is a pair of functions $f : I_A \rightarrow O_A$, $g : I_B \rightarrow O_B$ such that whenever Alice and Bob receive input pair (x, y) they respond with output pair $(f(x), g(y))$.

Given a game with a density $(\mathcal{G}, \pi)$ and a deterministic strategy as above, the probability of winning the game is

$$\sum_{x, y} \pi(x, y) V(f(x), g(y), x, y) = \sum_{(f(x), g(y), x, y) \in W} \pi(x, y).$$

The *classical value* or **deterministic value** of $(\mathcal{G}, \pi)$ is defined to be the supremum of this winning probability over all possible deterministic strategies. Computing this value often requires searching over all pairs of functions f, g and so can be computationally difficult. In fact, often it falls in the realm of NP-hard problems.

The classical value of a game is generally denoted $\omega(\mathcal{G}, \pi)$, but we shall write $\omega_{det}(\mathcal{G}, \pi)$, since we will be examining several different versions of the value of a game and want a notation that distinguishes them.

Here is a very simple example.

2.1. The CHSH Game. CHSH stands for Clauser-Horne-Shimony-Holt [CHSH69]. Here $I_A = I_B = O_A = O_B = \mathbb{Z}_2$ —the binary field. The function V is most easily described by saying that, given the input pair (x, y) , the players win if their output pair (a, b) satisfies

$$a + b = xy.$$

Let's say that they also know that the probability is given by $\pi(x, y) = 1/4$, i.e., the uniform distribution on the 4 possible input pairs.

If they choose the deterministic strategy of always returning 0 no matter what input they receive then $a + b = 0$ and so they will win unless the input pair was $(1, 1)$. Thus the expected value of this deterministic strategy is $3/4$.

It is not difficult to show that among all deterministic strategies, this is the one with the greatest expected value and that there is exactly one other deterministic strategy with value $3/4$.

If we fix $0 < t \leq 1$ and change the input probability to

$$\pi(0, 0) = \pi(1, 0) = \pi(0, 1) = t/3, \quad \pi(1, 1) = 1 - t,$$

then it is not hard to see that the best deterministic strategy depends on t . For $t < 1/2$,

$$\pi(1, 1) > \pi(0, 0) + \pi(0, 1) + \pi(1, 0),$$

and deterministic strategies that win for input pair $(1, 1)$ have a greater winning probability.

We should remark that many authors define a game as consisting of (I_A, I_B, O_A, O_B, V) and the probability density π on inputs. Because we want the freedom to vary the density on inputs we define a *game* as a 5-tuple,

$$\mathcal{G} = (I_A, I_B, O_A, O_B, V),$$

and a *game with density* as a 6-tuple $(\mathcal{G}, \pi)$.

2.2. Random Strategies, Perfect Strategies, and the Classical Value of a Game. The fact that these games are memoryless allows for the possibility of strategies that produce the answer pairs randomly, i.e., if Alice and Bob receive input pair (x, y) at two different rounds they might return two different output pairs.

A **random strategy** for a game yields a *conditional probability density*,

$$p(a, b|x, y), \quad x \in I_A, y \in I_B, a \in O_A, b \in O_B,$$

which gives the conditional probability that Alice and Bob return output pair (a, b) , given that they received input pair (x, y) .

Often in the physics literature, they are concerned with how one can set up an experiment that produces a given density $p(a, b|x, y)$, in which case, by a random strategy they mean the description of the experiment as opposed to the density itself. Other times a random strategy might be described, such as “first Alice flips a coin and Bob rolls a die, and then...”, without explicitly giving the density.

Since we will generally not be concerned with precisely how a given conditional probability density can be created, we will often conflate these terms, especially in the context of games, and often refer to a conditional probability density as a random strategy for the game.

A deterministic strategy $f : I_A \rightarrow O_A$, $g : I_B \rightarrow O_B$ corresponds to a density of the form

$$p(a, b|x, y) = \begin{cases} 1, & a = f(x), b = g(y) \\ 0, & \text{else} \end{cases}.$$

Note that any density such that $p(a, b|x, y) \in \{0, 1\}$ arises in such a manner from a deterministic strategy.

Given a game $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ with density on inputs π and a random strategy $p(a, b|x, y)$ the *expected value of winning the game* is given by

$$\mathbb{E}(p) = \sum_{a, b, x, y} \pi(x, y) V(a, b, x, y) p(a, b|x, y) = \sum_{(a, b, x, y) \in W} \pi(x, y) p(a, b|x, y).$$

Hence, the deterministic value of a game $\mathcal{G}$ with density π on inputs, $\omega_{det}(\mathcal{G}, \pi)$, is the supremum of $\mathbb{E}(p)$ over all densities of the above form. When we introduce other concepts of the value of a game, it will be by varying the sets of probability densities that we take the supremum over.

Thus, each game determines with prior distribution corresponds to a linear functional $\mathbb{E}(\cdot)$ on the set of all densities $p(a, b|x, y)$, but not every linear functional of the form

$$\mathbb{F}(p) = \sum_{a,b,x,y} c_{a,b,x,y} p(a, b|x, y),$$

arises from a game. One way that some authors generalize the definition of games is to allow for more general linear functionals—for example, by replacing the restriction that $V(a, b, x, y) \in \{0, 1\}$ by $V(a, b, x, y) \in [0, 1]$, which amounts to a probabilistic interpretation of “how correct” an answer is. Some authors study game biases, and allow functionals where the coefficients $c_{a,b,x,y}$ are in $[-1/2, +1/2]$.

Given a game, a density is called **perfect** if the probability of it returning an incorrect answer is 0, i.e.,

$$\forall x, y, V(a, b, x, y) = 0 \implies p(a, b|x, y) = 0.$$

Note that if p is perfect, then $\mathbb{E}(p) = 1$ no matter what the density is on inputs. Conversely, if a density on inputs has full support, i.e., $\pi(x, y) > 0, \forall x, y$, then any density with $\mathbb{E}(p) = 1$ is necessarily perfect.

Much research has focused on whether or not a given game $\mathcal{G}$ has a perfect density that belongs to particular families of densities. These various families of densities are the topic of the next section.

2.3. Local Densities. There are classical ways to produce conditional probability densities of the form $p(a, b|x, y)$ using probability spaces and random variables. However, there are other ways to produce such densities using measurements of quantum systems in entangled states. In this section we describe these various sets of densities and give their mathematical characterizations.

Suppose that we are given a *probability space* $(T, \mathcal{M}, m)$, i.e., a set T , a σ -algebra $\mathcal{M}$ of subsets, and a non-negative measure $m : \mathcal{M} \rightarrow \mathbb{R}^+$ satisfying $m(T) = 1$. Given finite input and output sets I_A, I_B, O_A, O_B let $f_x : T \rightarrow O_A, x \in I_A$ and $g_y : T \rightarrow O_B, y \in I_B$ be measurable functions. Such functions correspond to a choice of subsets, $E_{x,a} := \{t : f_x(t) = a\}, F_{y,b} := \{t : g_y(t) = b\}$ in $\mathcal{M}$ such that $\{E_{x,a} : a \in O_A\}$ and $\{F_{y,b} : b \in O_B\}$ are measurable partitions of T for all x, y . Then the density,

$$p(a, b|x, y) := m(\{t : f_x(t) = a, g_y(t) = b\}) = \int_T \chi_{E_{x,a}}(t) \chi_{F_{y,b}}(t) dm(t),$$

is called a *local* or *classical* density. We write C_{loc} to denote the set of all densities that can be obtained this way. Often for increased clarity we might

write $C_{loc}(n_A, n_B, k_A, k_B)$ where n_A, n_B, k_A, k_B are the respective cardinalities of the sets I_A, I_B, O_A, O_B .

Note that if we fix x, y, t , then $\chi_{E_{x,a}}(t)\chi_{F_{y,b}}(t)$ is 1 for exactly one pair (a, b) and 0 for all other pairs and so corresponds to the density of a deterministic strategy. It follows that the given local density $p(a, b|x, y)$, being the integral, is in the convex hull in the vector space $\mathbb{R}^{n_A n_B k_A k_B}$ of the densities corresponding to deterministic strategies. Since there are only a finite number of deterministic strategies, we see that C_{loc} is the convex polytope of the densities coming from deterministic strategies.

Note that if we are given a local density as above and a game $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ with density π , then

$$\mathbb{E}(p) = \int_T \sum_{x,y,a,b} \pi(x, y) V(f_x(t), g_y(t), x, y) dm(t).$$

The *local value* of $(\mathcal{G}, \pi)$, denoted $\omega_{loc}(\mathcal{G}, \pi)$ is defined to be the supremum of $\mathbb{E}(p)$ over all local densities p .

If we set $\hat{f}_t(x) = f_x(t)$ and $\hat{g}_t(y) = g_y(t)$, then for each $t \in T$, the pair $\hat{f}_t, \hat{g}_t$ is a deterministic strategy and $\mathbb{E}(p)$ is the average of the value of these deterministic strategies over $t \in T$.

Thus, $\omega_{det}(\mathcal{G}, \pi) = \omega_{loc}(\mathcal{G}, \pi)$.

The term *local* refers back to Einstein's theory of **local hidden variables**, which was one attempt to explain the type of randomness that occurred in quantum experiments [EPR35]. If Einstein's theory had been correct, then when two parties performed experiments in separated, physically isolated labs on a shared quantum state, any density that they would be able to produce would be local.

The mathematical physicist J.S. Bell was the first to make this argument [Bel64]. He explicitly described the polytope of local densities in the case that $n_A = n_B = k_A = k_B = 2$ and also described the strictly larger set of densities that one could obtain if one assumed the standard model for quantum entanglement. If we express this polytope as an intersection of $1/2$ -spaces, say

$$C_{loc}(n_A, n_B, k_A, k_B) = \cap_{i=1}^M \{(a, b, x, y) : L_i(a, b, x, y) \leq 1\},$$

then, up to some scaling these linear inequalities are called the *Bell inequalities*.

Later physicists were able to conduct experiments yielding densities that lay outside of Bell's polytope, i.e., violating one of the Bell inequalities, verifying that entanglement is an actual phenomena. One of the earliest such experiments/inequalities was due to [CHSH69] and is now known as the *CHSH inequality*.

However, as we will soon see, there are various potential mathematical models for describing quantum entanglement and deciding if these models

all yield the same sets of densities or yield distinct sets of densities is one place where the theory of these games has been extremely effective.

2.4. Multiplayer Games and Proof Verification. Although we will focus on two player games, there is a well-developed theory of cooperative games with a larger number of players, along with applications using games as a probabilistic means of checking proofs. See [ALM⁺98, AS98, BFL91, BGKW88, CWY15].

3. MODELS FOR QUANTUM CORRELATIONS: TSIRELSON'S PROBLEMS

In the last section we mentioned that there were different models for quantum densities without really addressing what these models are. We remedy that problem here.

To motivate our discussions we need to first describe, in an axiomatic manner, some of the foundations of quantum mechanics. Each quantum mechanical system has a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle$ affiliated with the system such that the unit vectors in this space are said to represent the *pure states* of the system. We adopt the physicists' convention that our inner products are conjugate linear in the first variable and linear in the second variable. If one wishes to conduct a quantum measurement that has k outcomes, then there will be a set of k positive operators, $P_1, \dots, P_k$ satisfying $\sum_{a=1}^k P_a = I_{\mathcal{H}}$ with the property that if the system is in the pure state $\psi \in \mathcal{H}$, then the probability of observing outcome a , $1 \leq a \leq k$ is given by

$$\langle \psi | P_a | \psi \rangle.$$

Such a set of operators is called a *positive operator-valued measure*, or k -POVM, for short.

By an (n, k) -POVM, we mean a collection $\{P_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$, where for each fixed x , the set $\{P_{x,a} : 1 \leq a \leq k\}$ is a k -POVM.

Suppose that Alice and Bob have separated, isolated labs and they can each perform one of n_A , respectively, n_B , quantum measurements and each measurement has, respectively, k_A and k_B outcomes. We let $p(a, b | x, y)$ denote the conditional probability density that Alice gets outcome a and Bob gets outcome b , when they perform measurements x and y , respectively. Such densities are also called *quantum correlations* and Tsirelson was interested in mathematical descriptions of the set of all such conditional densities.

It turns out that the axiomatic quantum theory allows for several possible mathematical descriptions of these sets of densities and Tsirelson was interested in whether these were all the same [Tsi93]. So we start with the possible descriptions.

The basic quantum model assumes that Alice's and Bob's labs are described by finite dimensional state spaces, $\mathcal{H}_A, \mathcal{H}_B$ and that the state of their combined labs is given by a unit vector $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Alice's and Bob's measurements are each given by an (n, k) -POVM, $\{E_{x,a} : 1 \leq x \leq$

$n_A, 1 \leq a \leq k_A\}$ and $\{F_{y,b} : 1 \leq y \leq n_B, 1 \leq b \leq k_B\}$, which means we have families of positive operators such that

$$\sum_{a=1}^{k_A} E_{x,a} = I_{\mathcal{H}_A}, \forall x, \text{ and } \sum_{b=1}^{k_B} F_{y,b} = I_{\mathcal{H}_B}, \forall y,$$

and

$$p(a, b|x, y) = \langle \psi | (E_{x,a} \otimes F_{y,b}) \psi \rangle.$$

We let C_q , or for definiteness, $C_q(n_A, n_B, k_A, k_B)$ denote the set of all $p(a, b|x, y)$ that can be obtained as above, which we call the *quantum correlations* or *quantum densities*. Note that since $0 \leq p(a, b|x, y) \leq 1$ that we can always regard $C_q(n_A, n_B, k_A, k_B)$ as a subset of the compact set $[0, 1]^{n_A n_B k_A k_B}$. Generally, we shall be interested in the case that $n_A = n_B = n$ and $k_A = k_B = k$, in which case we shorten this to $C_q(n, k)$.

A slightly more general model allows $\mathcal{H}_A$ and $\mathcal{H}_B$ to be arbitrary Hilbert spaces in which case we denote this larger set by $C_{qs}(n_A, n_B, k_A, k_B)$ where the subscript stands for *quantum spatial*. In this case we are assuming that the unit vector ψ belongs not just to the algebraic tensor product of the two spaces, but in the completion of the algebraic tensor product in the inner product, defined on elementary tensors by

$$\langle \psi_1 \otimes \phi_1 | \psi_2 \otimes \phi_2 \rangle = \langle \psi_1 | \psi_2 \rangle_{\mathcal{H}_A} \cdot \langle \phi_1 | \phi_2 \rangle_{\mathcal{H}_B}.$$

There is no reason that either of these sets needs to be closed. However, a nice result that uses the theory of *residually finite dimensional (RFD)* C*-algebras is that they both have the same closure and we set

$$C_{qa}(n_A, n_B, k_A, k_B) := C_q(n_A, n_B, k_A, k_B)^- = C_{qs}(n_A, n_B, k_A, k_B)^-.$$

These are called the *quantum approximate* correlations. For a proof of this closure result see [JNP⁺11] or [Fri12].

An even more general model is to assume that the combined state space of Alice and Bob does not decompose as a tensor product but instead that it is a single Hilbert space $\mathcal{H}$ so that they each have POVM's on this space,

$$\begin{aligned} \{E_{x,a} : 1 \leq x \leq n_A, 1 \leq a \leq k_A\} &\subseteq B(\mathcal{H}), \\ \{F_{y,b} : 1 \leq y \leq n_B, 1 \leq b \leq k_B\} &\subseteq B(\mathcal{H}), \end{aligned}$$

with the property that $E_{x,a} F_{y,b} = F_{y,b} E_{x,a}$, $\forall x, y, a, b$. We call this a *commuting model*. There is no requirement that Alice's, or Bob's, operators commute among themselves for different inputs.

The set of all

$$p(a, b|x, y) = \langle \phi | E_{x,a} F_{y,b} \phi \rangle,$$

that can be obtained in this manner for some commuting model and some unit vector ϕ is denoted $C_{qc}(n_A, n_B, k_A, k_B)$ and is called the *quantum commuting* correlations. This set is known to be closed but the proof needs some C*-algebra theory. See [JNP⁺11].

The explanation for this commuting hypothesis is that the outcome should not depend on the order of applying their measurements. Note that in the tensor cases we have that

$$E_{x,a} \otimes F_{y,b} = (E_{x,a} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes F_{y,b}) = (I_{\mathcal{H}_A} \otimes F_{y,b})(E_{x,a} \otimes I_{\mathcal{H}_B}).$$

so it is a commuting correlation.

It is known that

$$C_q(n_A, n_B, k_A, k_B) \subseteq C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{qa}(n_A, n_B, k_A, k_B) \subseteq C_{qc}(n_A, n_B, k_A, k_B).$$

When people refer to [quantum correlations](#), they mean one of these sets. The work of J.S. Bell [Bel64], showed that $C_{loc}(2, 2, 2, 2) \neq C_q(2, 2, 2, 2)$. Later experimentalists were able to produce quantum correlations that belonged to this larger set giving solid evidence that entanglement is an actual phenomena that can not be explained by Einstein's local hidden variable theory.

In the case that $n_A = n_B = k_A = k_B = 2$, Tsirelson [Tsi93] proved that these various sets of quantum correlations are all equal, and wondered if this could be true more generally. The equality/nonequality of these various pairs of sets became known as the [Tsirelson problems](#).

Operator algebraists became interested in the Tsirelson problems, when the combined work of [JNP⁺11] and [Oza13] showed that the *Connes Embedding Problem*, had an affirmative answer if and only if

$$C_{qa}(n_A, n_B, k_A, k_B) = C_{qc}(n_A, n_B, k_A, k_B),$$

for all n_A, n_B, k_A, k_B . See [Fri12] for another proof of the results in [JNP⁺11]. We explain this more fully later.

Thanks largely to the theory of non-local games, and studying whether or not games have perfect strategies in each of these set, we now know that all of these sets are distinct when n_A, n_B, k_A, k_B are sufficiently large.

The first breakthrough was when Slofstra[Slo19a] created a game with a perfect strategy in C_{qa} but not in C_q , thus showing that C_q is not closed. Later, [KPS18] would show that Slofstra's result implied that C_{qs} is also not closed. Coladangelo and Stark[CS18] created a game with a perfect strategy in C_{qs} but not in C_q , giving another proof that C_q is not closed.

The greatest breakthrough was when Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen [JNV⁺20] proved that $C_{qa} \neq C_{qc}$, by constructing a game that has a perfect strategy in C_{qc} , but no perfect strategy in C_{qa} . Much of their work involves implicit steps, so very little is known about how large n_A, n_B, k_A, k_B must be for these sets to be separated, and the game is also only given implicitly.

Questions still remain about equality/nonequality of these sets for small values. For example, the work of [DPP19] shows that $C_q(n_A, n_B, k_A, k_B)$ is not closed for any $n_A, n_B \geq 5$, $k_A, k_B \geq 2$, while the work of Tsirelson

and Bell showed that $C_q(2, 2, 2, 2)$ is closed. Coladangelo [Col20] shows $C_q(4, 4, 3, 3)$ is not closed.

It is believed that $C_q(3, 3, 2, 2)$ is not closed and there is even a functional arising from Bell's work, called the *I3322 equation* that is believed to not attain its supremum on this set. There is a considerable literature on this problem. See [PV10] for an introduction to this literature.

For each of the linear functionals defining the polytope $C_{loc}(n_A, n_B, k_A, k_B)$ it is interesting to know how much larger its supremum can be over the set $C_q(n_A, n_B, k_A, k_B)$ than over C_{loc} . There is considerable research on such problems often under the heading, *maximal violations of Bell's inequalities*.

The work of [DPP19] shows that the geometry of the convex set $C_q(5, 5, 2, 2)$ is quite intricate.

Theorem 3.1 ([DPP19]). *Let $\frac{\sqrt{5}-1}{2\sqrt{5}} \leq t \leq \frac{\sqrt{5}+1}{2\sqrt{5}}$ and for $0 \leq a, b \leq 1$, $1 \leq x, y \leq 5$ set*

$$p(0, 0|x, x) = t, \quad p(0, 1|x, x) = p(1, 0|x, x) = 0, \quad p(1, 1|x, x) = 1 - t,$$

and for $x \neq y$, set

$$p(0, 0|x, y) = \frac{1}{4}t(5t - 1), \quad p(0, 1|x, y) = p(1, 0|x, y) = \frac{5}{4}t(1 - t),$$

$$p(1, 1|x, y) = \frac{1}{4}(1 - t)(4 - 5t).$$

Then $p \in C_{qa}(5, 5, 2, 2)$ for all t in this interval, but $p \in C_q(5, 2)$ only for t rational.

Although all of these mathematical models for quantum correlations are now known to be distinct, there is, currently, not much agreement on which model is more correct for physics.

3.1. Nonsignalling Densities. Any set of nonnegative numbers $p(a, b|x, y)$, $1 \leq x \leq n_A$, $1 \leq y \leq n_B$, $1 \leq a \leq k_A$, $1 \leq b \leq k_B$, satisfying $\sum_{a,b} p(a, b|x, y) = 1$, $\forall x, y$ can be thought of as a conditional probability density. Such a density is called **nonsignalling** provided that

$$\sum_b p(a, b|x, y_1) = \sum_b p(a, b|x, y_2) \text{ and } \sum_a p(a, b|x_1, y) = \sum_a p(a, b|x_2, y),$$

for all x, x_1, x_2, y, y_1, y_2 . In this case there are well defined *marginal densities* given by

$$p_A(a|x) := \sum_b p(a, b|x, y) \text{ and } p_B(b|y) := \sum_a p(a, b|x, y).$$

The set of all such nonsignalling densities is denoted $C_{ns}(n_A, n_B, k_A, k_B)$ and is clearly a closed set.

The reason for the name comes from the fact that if, instead, the marginal density $\sum_b p(a, b|x, y)$ depended on y , then Alice can obtain some information about which input y Bob has been given, violating the nonsignalling hypothesis.

It is straightforward to verify that all of the quantum correlations defined above are nonsignalling. In fact, in the quantum commuting model one has

$$p_A(a|x) = \langle \psi | E_{x,a} \psi \rangle \text{ and } p_B(b|y) = \langle \psi | F_{y,b} \psi \rangle.$$

Thus, $C_{qc}(n_A, n_B, k_A, k_B) \subseteq C_{ns}(n_A, n_B, k_A, k_B)$.

There are also games that have perfect strategies in C_{ns} and no perfect strategy in C_{qc} , showing that this containment is also strict. For example, [CNM⁺07] gives examples of games arising from graph coloring that show this distinction.

There is quite a bit of interest in studying perfect nonsignalling strategies for games, see [BHTT23b, LMP⁺20, TT24] for recent work and references to the earlier work in this area.

3.2. Quantum Values of Games. Given a game $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ along with a prior density π on inputs, for $t = loc, q, qs, qa, qc, ns$ we define the corresponding value of the game to be

$$\omega_t(\mathcal{G}, \pi) = \sup_{p \in C_t(n_A, n_B, k_A, k_B)} \sum_{a,b,x,y} \pi(x, y) V(a, b, x, y) p(a, b|x, y).$$

By the fact that the closure of C_q and C_{qs} are both C_{qa} and the various containments, we have that

$$\omega_{det}(\mathcal{G}, \pi) = \omega_{loc}(\mathcal{G}, \pi) \leq \omega_q(\mathcal{G}, \pi) = \omega_{qs}(\mathcal{G}, \pi) = \omega_{qa}(\mathcal{G}, \pi) \leq \omega_{qc}(\mathcal{G}, \pi) \leq \omega_{ns}(\mathcal{G}, \pi).$$

The term *nonlocal game* was first used in [CHTW04]. It refers to the fact that many games can have larger values if we allow any of these nonlocal densities to be used as strategies. In some ways the term is shorthand for *games played with nonlocal strategies*.

The CHSH game $\mathcal{G}$ with uniform density π is one such example. In fact,

$$\omega_{loc}(\mathcal{G}, \pi) = 3/4 < \omega_q(\mathcal{G}, \pi) = \cos^2(\pi/8).$$

See [CHTW04] for a proof. In fact, the CHSH game was reverse engineered so that its expected value functional, $L(a, b, x, y) = \mathbb{E}(p)$ is the linear equation in the famous CHSH inequality.

3.3. POVM's versus PVM's. An (n, k) -POVM $\{P_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ is called an (n, k) -PVM provided that every operator $P_{x,a}$ is not just positive but is a self-adjoint projection, i.e., $P_{x,a} = P_{x,a}^* = P_{x,a}^2$.

Note that since these are, currently, operators on a Hilbert space, we also have that $P_{x,a}P_{x,b} = 0, \forall a \neq b$. This is not the case for abstract $*$ -algebras, a point that we shall return to later.

In [JNP⁺11] and in [Fri12] it is shown that each of the sets C_q, C_{qs}, C_{qa} , and C_{qc} remain unchanged if in their definitions (n, k) -PVM's are used instead of (n, k) -POVM's. The fact that C_{loc} remains unchanged follows from the fact that characteristic functions of sets are projections in an abelian algebra.

In the case of the tensor product models, this fact follows from a technique used in mathematical physics called *state purification* or from the operator

algebra technique of dilating a positive contraction to a projection. But the proof for the case of C_{qc} is more sophisticated and relies on Boca's theorem [Boc91] on free products of completely positive maps. Recently, a much simpler proof of Boca's result has been given [DK17], which makes the C_{qc} case more transparent.

In the remainder of this chapter, we shall use (n, k) -PVM's in the definitions of quantum correlations. Note that in the case of local densities, the characteristic functions of sets are projections in the abelian algebra of functions, so local densities were already defined in terms of abelian (n, k) -PVM's.

3.4. Densities and States on Tensor Products of Free Group Algebras. Given a (discrete) group G let $\mathbb{C}(G)$ denote the free vector space with basis $\{u_g : g \in G\}$. Defining a product on $\mathbb{C}(G)$ by setting $u_g \cdot u_h = u_{gh}$ makes $\mathbb{C}(G)$ into an algebra. Alternatively, one may think of $\mathbb{C}(G)$ as finitely supported functions on G and define the multiplication by *convolution*. Setting $u_g^* = u_{g^{-1}}$ and extending conjugate linearly, gives $\mathbb{C}(G)$ the structure of a $*$ -algebra. This $*$ -algebra is universal for unitary representations of G . That is, given a Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and $\mathcal{U}(\mathcal{H})$ the group of unitary operators. Then $\rho : G \rightarrow \mathcal{U}(\mathcal{H})$ is a group homomorphism if and only if there is a unital $*$ -homomorphism $\hat{\rho} : \mathbb{C}(G) \rightarrow B(\mathcal{H})$ satisfying $\hat{\rho}(u_g) = \rho(g)$ for all $g \in G$.

Briefly, we recall that to form the full group C^* -algebra $C^*(G)$, one first defines a norm on $\mathbb{C}(G)$ by setting

$$\|w\| = \sup_{\rho} \|\hat{\rho}(w)\|,$$

where the supremum is over all unitary representations of G on a Hilbert space, and then completes in this norm.

We also very briefly recall the tensor theory of C^* -algebras. For more details see [BO08] or [Con00]. Given two C^* -algebras, $\mathcal{A}, \mathcal{B}$ there are generally many norms that can be put on their algebraic tensor product so that the completion is a C^* -algebra. Among these many possible norms, there is a unique minimal norm and that completion yields the C^* -algebra denoted $\mathcal{A} \otimes_{min} \mathcal{B}$.

By a theorem of Takesaki (see [BO08]), if we represent $\mathcal{A} \subseteq B(\mathcal{H})$ and $\mathcal{B} \subseteq B(\mathcal{K})$, as operators on Hilbert spaces, then we can regard $\mathcal{A} \otimes \mathcal{B} \subseteq B(\mathcal{H} \otimes \mathcal{K})$ and the completion of this algebra of operators is $\mathcal{A} \otimes_{min} \mathcal{B}$. In particular, Takesaki's theorem tells us that this norm is independent of the particular representations.

Given a Hilbert space $\mathcal{L}$ and a pair of $*$ -homomorphisms $\pi : \mathcal{A} \rightarrow B(\mathcal{L})$ and $\rho : \mathcal{B} \rightarrow B(\mathcal{L})$ such that the set $\pi(\mathcal{A})$ commutes with the set $\rho(\mathcal{B})$, one obtains a $*$ -homomorphism $\pi \odot \rho : \mathcal{A} \otimes \mathcal{B} \rightarrow B(\mathcal{L})$ by setting $\pi \odot \rho(a \otimes b) = \pi(a)\rho(b)$ and extending linearly. The max norm of an element $u \in \mathcal{A} \otimes \mathcal{B}$ is defined as the supremum of $\|\pi \odot \rho(u)\|$ over all Hilbert spaces $\mathcal{L}$ and all

such pairs π, ρ . The C^* -algebra $\mathcal{A} \otimes_{\max} \mathcal{B}$ is the completion of $\mathcal{A} \otimes \mathcal{B}$ in this norm.

In the case of two groups, G, H we have that $\mathbb{C}(G) \otimes \mathbb{C}(H) \simeq \mathbb{C}(G \times H)$ and the norm $C^*(G) \otimes_{\max} C^*(H) \simeq C^*(G \times H)$, i.e., it is the norm that comes from all considering all unitary representations of $G \times H$. While the norm $C^*(G) \otimes_{\min} C^*(H)$ is the norm that comes from considering the spatial tensor products of pairs of unitary representations, $\pi : G \rightarrow B(\mathcal{H})$ and $\rho : H \rightarrow B(\mathcal{K})$.

If we let $\mathbb{F}(n, k)$ denote the free product of n copies of the cyclic group of order k , then this is a group with generators $u_x, 1 \leq x \leq n$ satisfying $u_x^k = 1$ and representations of this group are in one-to-one correspondence with a choice of n unitaries $U_x \in \mathcal{U}(\mathcal{H})$ satisfying $U_x^k = I_{\mathcal{H}}$ for all x .

As a $*$ -algebra $\mathbb{C}(\mathbb{F}(n, k))$ is the free $*$ -algebra with generators u_x satisfying $u_x^k = 1$ for all x .

If we set $\zeta = e^{2\pi i/k}$ and set

$$e_{x,a} = \frac{1}{k} \sum_{j=1}^k (\zeta^{-a} u_x)^j,$$

then these are the corresponding spectral projections for u_x and

$$u_x = \sum_{j=1}^k \zeta^j e_{x,j}.$$

The set $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ also generate $\mathbb{C}(\mathbb{F}(n, k))$ and this group $*$ -algebra can also be seen to be the free unital $*$ -algebra with these generators and relations:

- $e_{x,a} = e_{x,a}^* = e_{x,a}^2$,
- $\sum_{a=1}^k e_{x,a} = 1$,
- $e_{x,a} e_{x,b} = 0, \forall a \neq b$.

It is perhaps worth mentioning that this last relation is really necessary because in a general $*$ -algebra, a set of projections that sum to the identity need not be orthogonal, except when $k \leq 3$ [ST02].

We see that (n, k) -PVM's correspond to $*$ -representations of $\mathbb{C}(\mathbb{F}(n, k))$, or equivalently, of $C^*(\mathbb{F}(n, k))$.

If we regard the generators of $\mathbb{C}(\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B))$ as $\{e_{x,a}, f_{y,b} : 1 \leq x \leq n_A, 1 \leq a \leq k_A, 1 \leq y \leq n_B, 1 \leq b \leq k_B\}$, then

$$\mathbb{C}(\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B)) = \mathbb{C}(\mathbb{F}(n_A, k_A)) \otimes \mathbb{C}(\mathbb{F}(n_B, k_B)),$$

via the identification $e_{x,a} f_{y,b} = e_{x,a} \otimes f_{y,b}$. In this setting the quantum correlations have the following characterizations.

Recall that a *state* s on a unital $*$ -algebra is a linear functional that is positive (that is, $s(w^*w) \geq 0$ for all w), and unital ($s(1) = 1$). The work of [JNP⁺11] and [Fri12] can be summarized as follows.

Theorem 3.2 ([JNP⁺11], [Fri12]). (1) $p(a, b|x, y) \in C_{qs}(n_A, n_B, k_A, k_B)$ if and only if there exist Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, unitary representations $\rho_A : \mathbb{F}(n_A, k_A) \rightarrow \mathcal{U}(\mathcal{H}_A)$, $\rho_B : \mathbb{F}(n_B, k_B) \rightarrow \mathcal{U}(\mathcal{H}_B)$ and a unit vector ψ in the completed tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $p(a, b|x, y) = \langle \psi | (\widehat{\rho}_A(e_{x,a}) \otimes \widehat{\rho}_B(f_{y,b})) \psi \rangle$.

(2) $p(a, b|x, y) \in C_q(n_A, n_B, k_A, k_B)$ if and only if there exists a unitary representation $\rho : \mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B) \rightarrow \mathcal{U}(\mathcal{H})$ with $\mathcal{H}$ finite dimensional and a unit vector ψ such that $p(a, b|x, y) = \langle \psi | \widehat{\rho}(e_{x,a} f_{y,b}) \psi \rangle$.

(3) $p(a, b|x, y) \in C_{qa}(n_A, n_B, k_A, k_B)$ if and only if there is a state s on the minimal C^* -tensor product $C^*(\mathbb{F}(n_A, k_A)) \otimes_{\min} C^*(\mathbb{F}(n_B, k_B))$ such that $p(a, b|x, y) = s(e_{x,a} \otimes f_{y,b})$.

(4) $p(a, b|x, y) \in C_{qc}(n_A, n_B, k_A, k_B)$ if and only if there is a state s on the maximal C^* -tensor product $C^*(\mathbb{F}(n_A, k_A)) \otimes_{\max} C^*(\mathbb{F}(n_B, k_B))$ such that $p(a, b|x, y) = s(e_{x,a} \otimes f_{y,b})$.

Alternatively, (2) can be replaced by the statement of (1) along with the additional requirement that both Hilbert spaces are finite dimensional. Also, (2) can be replaced by the same statement as (3), but with the extra requirement that the Gelfand-Naimark-Segal representation of the state is finite dimensional.

Also, (4) can be replaced by the same statement as (2), after removing the requirement that $\mathcal{H}$ be finite dimensional.

Thus, (1), (2), and (4) can all be stated in terms of different types of representations of $\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B)$.

The reason that the introduction of tensor products is used for (3) is that it is defined as limits of densities from (2) and while limits of states are again states, there is no good way to take limits of finite dimensional representations of algebras and obtain a representation, without first passing to the state induced by the representation.

Interestingly, while (2), (3) and (4) can be described in terms of states, there is no natural way to describe (1) in terms of states on the minimal tensor product.

For an interesting example of this difficulty, consider $C([0, 1]) \otimes_{\min} C([0, 1]) \simeq C([0, 1]^2)$. States on this C^* -algebra are in one-to-one correspondence with positive regular Borel probability measures on the unit square. We leave it to the reader to determine which measures can be represented on a tensor product of Hilbert spaces as in (1).

So the statement of the above theorem is rather fragmented since 3 of the 4 sets can be naturally described in terms of representations, while a different set of 3 of the 4 can be naturally represented in terms of states.

Thus, the different types of quantum correlations can be regarded as the study of the images of different types of states when restricted to the generators of the $*$ -algebra of the group $\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B)$.

In this context, Tsirelson's result on the equality of the various models for quantum correlations for the case $n_A = k_A = n_B = k_B = 2$ can be deduced

from the fact that the irreducible representations of the group $\mathbb{F}(2, 2) \times \mathbb{F}(2, 2)$ are at most 4 dimensional. However, Tsirelson's proof shows considerably more as we will see in the section on XOR games.

If instead of looking at states, we consider linear functionals s on $\mathbb{C}(\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B))$ that are positive on w^*w for w a linear combination of words in the generators of some bounded degree d and consider the set of all $p(a, b|x, y) = s(e_{x,a} \otimes f_{y,b})$ obtained in this fashion, then one obtains a decreasing sequence of sets whose intersection is $C_{qc}(n_A, n_B, k_A, k_B)$. Moreover, each of the sets obtained in this fashion can be shown to be a spectrahedron. This, decreasing sequence of spectrahedra is referred to as the *NPA hierarchy* and was introduced and studied in [NPA08].

3.5. The Problems of Connes and Kirchberg. Connes' Embedding Problem (CEP) [Con76] arose from the problem of trying to classify all von Neumann algebras that are type II_1 -factors. As the name suggests it is a question about whether or not type II_1 -factors embed into a particular II_1 -factor, described as an *ultrapower of the hyperfinite II_1 -factor*, and denoted $\mathcal{R}^\omega$. The CEP can equally well be restated as an embedding problem for separable tracial C^* -algebras, which is the approach that we shall present here in order to avoid introducing the theory of von Neumann algebras.

Given a unital $*$ -algebra $\mathcal{A}$ we call a linear functional $\tau : \mathcal{A} \rightarrow \mathbb{C}$ a **tracial state** provided that it is a state ($\tau(x^*x) \geq 0$ for all $x \in \mathcal{A}$ and $\tau(1) = 1$) and $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{A}$. Note that this last condition is equivalent to the requirement that τ vanishes on the linear span of the set of all $xy - yx$, which we call *commutators*. A state is called **faithful** provided that $\tau(x^*x) = 0 \implies x = 0$.

The matrix algebra M_n has a unique tracial state, which we denote by tr_n . This tracial state is given by

$$tr_n((a_{i,j})) = \frac{1}{n} \sum_{j=1}^n a_{j,j}.$$

There are many ways to describe the algebra $\mathcal{R}^\omega$. We present the way that is best for our purposes.

A *non-principal ultrafilter limit* is a non-zero multiplicative linear functional $\omega : \ell^\infty(\mathbb{N}) \rightarrow \mathbb{C}$ with the property that $\omega((a_n)) = \lim_n a_n$ whenever the limit exists. We set

$$\lim_\omega a_n := \omega((a_n)).$$

The name comes from the fact that they all arise from *non-principal ultrafilters on $\mathbb{N}$* . From a functional analysis perspective, non-principal ultrafilter limits exist by the Gelfand-Naimark theorem applied to the abelian Banach algebra $\ell^\infty(\mathbb{N})$. See [Con00] for more.

Note that the ℓ^∞ -direct sum of the matrix algebras,

$$\mathcal{A} := \mathbb{C} \oplus M_2 \oplus M_3 \oplus \cdots := \{A = A_1 \oplus A_2 \oplus \cdots : \sup_n \|A_n\| < +\infty\},$$

is a C*-algebra with product

$$A \cdot B := A_1 B_1 \oplus A_2 B_2 \oplus \cdots .$$

Given a non-principal ultrafilter ω on $\mathbb{N}$, the set

$$\mathcal{I}_\omega := \{A = A_1 \oplus A_2 \oplus \cdots : \lim_{\omega} \text{tr}_n(A_n^* A_n) = 0\},$$

is a closed two-sided ideal and hence there is a quotient C*-algebra denoted

$$\mathcal{R}^\omega := \mathcal{A} / \mathcal{I}_\omega.$$

Setting

$$\tau_\omega(A + \mathcal{I}_\omega) = \lim_{\omega} \text{tr}_n(A_n),$$

gives a well-defined tracial state on $\mathcal{R}^\omega$. For more information, see [BO08].

Interestingly, if we assume the continuum hypothesis, then for any two non-principal ultrafilters, these algebras are *-isomorphic. But without the continuum hypothesis there are many distinct isomorphism classes. For details see [CL15].

For those not familiar with ultrapower constructions, we would like to say a few imprecise words about them, to provide intuition. Saying that something “happens” in $\mathcal{R}^\omega$ is equivalent to saying that it “almost happens in matrix algebras with ever increasing accuracy”. For example, in group representation theory there is the notion of a *hyperlinear* group. These are groups with a sequence of maps into the group of unitary matrices of increasing size that are in a certain sense becoming multiplicative in the limit. Alternatively, these are more simply described as the groups for which there exists a one-to-one homomorphism from the group into the unitary group of $\mathcal{R}^\omega$. Voiculescu’s theory of *matricial microstates* is another example where a statement about something almost happening in matrix algebras is equivalent to an exact statement in $\mathcal{R}^\omega$ and there is a conjecture about matricial microstates that is equivalent to Connes’ Embedding Problem, see [DP16] for a discussion.

Connes’ Embedding Problem asks if given a II_1 -factor $(\mathcal{M}, \tau)$ with a separable predual, does there always exist a *-homomorphism from $\mathcal{M}$ into $\mathcal{R}^\omega$. Because the trace is always unique on a II_1 -factor, this *-homomorphism must in fact be trace preserving. By the general properties of ultrapowers mentioned above, this version of CEP asks if the trace τ “happens” in $\mathcal{R}^\omega$, and so not surprisingly, turns out to be equivalent to asking if τ can be approximated in a certain sense by the trace on matrix algebras.

To avoid detailed discussions of von Neumann algebras, we state an equivalent reformulation in terms of C*-algebras.

Connes’ Embedding Problem: Does every separable, tracial C*-algebra $(\mathcal{A}, \tau)$ have a trace preserving *-homomorphism into $(\mathcal{R}^\omega, \tau_\omega)$? That is, does there always exist a unital *-homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{R}^\omega$ such that $\tau(a) = \tau_\omega(\pi(a))$, $\forall a \in \mathcal{A}$?

The equivalence of this stated version of CEP and the version involving II_1 -factors is not difficult for those with a background in von Neumann algebras. The proof uses standard methods of passing between C^* -algebras and von Neumann algebras. One starts by taking the Gelfand-Naimark-Segal representation of $(\mathcal{A}, \tau)$ on a Hilbert space, taking the double commutant and showing that it is a von Neumann algebra with a trace.

If $(\mathcal{A}, \tau)$ has a trace preserving $*$ -homomorphism into $(\mathcal{R}^\omega, \tau_\omega)$ for some non-principal ultrafilter ω , then it has a trace preserving $*$ -homomorphism into $(\mathcal{R}^\beta, \tau_\beta)$ for all non-principal ultrafilters β , so whether or not the CEP is true or false is independent of the continuum hypothesis. See [CL15].

Recall that a trace is called *faithful* provided that $\tau(x^*x) = 0 \implies x = 0$. It is enough to consider C^* -algebras and faithful traces in the above and when the trace is faithful such a π is necessarily one-to-one. This fact justifies the term “embedding” in Connes’ embedding problem.

Because there is a unique trace on a II_1 -factor, the Connes’ Embedding Problem is equivalent to requiring that for every separable, C^* -algebra $\mathcal{A}$ with a faithful trace there is a $*$ -homomorphism (not necessarily trace preserving) into $\mathcal{R}^\omega$.

Kirchberg studied the question of whether or not there was a unique C^* -norm on the product of two free groups. If we let $\mathbb{F}_n$ denote the free product of n copies of the infinite cyclic group $(\mathbb{Z}, +)$, then he proved the following:

Theorem 3.3 ([Kir94]). *The following are equivalent:*

- (1) *Connes’ Embedding Problem has a positive answer,*
- (2) $C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n), \forall n \geq 2,$
- (3) *there exists $n \geq 2$ such that $C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n).$*

We shall refer to the equality of these tensor norms as **Kirchberg’s Problem**, so that Kirchberg’s Problem and Connes’ Embedding Problem are equivalent by Kirchberg’s theorem.

Once one knows this equivalence for the product of these free groups, it is not difficult to see that one can use other free groups as well—in particular, products of the groups of the form $\mathbb{F}(n, k)$. See [FKPT18, Proposition 3.3].

This leads us to the relation between Kirchberg’s problem and the sets C_{qa} and C_{qc} .

Theorem 3.4 ([JNP⁺11]). *If Kirchberg’s Problem has an affirmative answer, then*

$$C_{qa}(n_A, n_B, k_A, k_B) = C_{qc}(n_A, n_B, k_A, k_B), \forall n_A, n_B, k_A, k_B.$$

The converse is due to Ozawa:

Theorem 3.5 ([Oza13]). *Kirchberg’s Problem has an affirmative answer if and only if*

$$C_{qa}(n_A, n_B, k_A, k_B) = C_{qc}(n_A, n_B, k_A, k_B), \forall n_A, n_B, k_A, k_B.$$

In [JNV⁺20] it was shown that Kirchberg's Problem, and hence Connes' Embedding Problem, both have negative answers by producing a game with a perfect strategy in C_{qc} but no perfect strategy in C_{qa} . Note that for this negative result, Ozawa's result is not necessary only [JNP⁺11] is needed.

One key point is that [JNV⁺20] originally refuted Tsirelson's Problem and hence, by [JNP⁺11], Kirchberg's Problem, and it is only the deep and lengthy equivalence between Kirchberg's Problem and Connes' Embedding Problem (CEP) that made it a refutation of CEP.

Later, when we discuss synchronous games and synchronous correlations, we will see a more direct refutation of CEP, that does not use its equivalence to Kirchberg's Problem.

3.6. Algebraic Conditions for Perfect qc-Strategies. A. Bene Watts, J. W. Helton, and I. Klep [BHK23] have given purely algebraic conditions that determine if a game $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ has a perfect qc-strategy.

Theorem 3.6. [BHK23, Theorem 1.1] *Let $\mathcal{F} = \{e_{x,a}f_{y,b} : V(a,b,x,y) = 0\} \subseteq \mathbb{C}(\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B))$; let $\mathcal{L}(\mathcal{F})$ denote the left ideal generated by $\mathcal{F}$; let $\mathcal{R}(\mathcal{F})$ denote the right ideal generated by $\mathcal{F}$; and let SOS denote the cone generated by all elements of the form w^*w . Then $\mathcal{G}$ has a perfect qc-strategy if and only if*

$$-1 \notin \mathcal{L}(\mathcal{F}) + \mathcal{R}(\mathcal{F}) + SOS.$$

The notation SOS is short for *sum of squares*. Given a general $*$ -algebra $\mathcal{A}$, the SOS cone is often denoted $\mathcal{A}^+$ and called the *positive cone*.

The method of proof of the above theorem is similar to the considerations in the *NPA hierarchy* [NPA08], in that both involve restricting the degrees of words in the generators.

3.7. Multipartite Correlations and Real Versus Complex Models.

There are similar definitions of these sets of conditional densities for multipartite situations. For example, in the tripartite case, $C_q(n_A, n_B, n_C, k_A, k_B, k_C)$ refers to the densities of the form

$$p(a, b, c | x, y, z) = \langle \phi | (E_{x,a} \otimes F_{y,b} \otimes G_{z,c}) \phi \rangle,$$

where ϕ is a unit vector in the tensor product of three Hilbert spaces, $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Densities in C_{qc} arise from a single Hilbert space on which all the operators act and satisfy,

$$E_{x,a}F_{y,b} = F_{y,b}E_{x,a}, E_{x,a}G_{z,c} = G_{z,c}E_{x,a}, F_{y,b}G_{z,c} = G_{z,c}F_{y,b}.$$

As with the bipartite case the densities that one obtains are the same if one insists that the operators come from a POVM or a PVM. The densities in C_{qc} can also be realized as the images of states on the group $\mathbb{F}(n_A, k_A) \times \mathbb{F}(n_B, k_B) \times \mathbb{F}(n_C, k_C)$.

Remarkably, the set of bipartite densities $C_q(n_A, n_B, k_A, k_B)$ remains unchanged if one insists that the underlying Hilbert spaces are real [RTW⁺21]. However, as soon as one passes to the tripartite case, then the requirement

that the three Hilbert spaces all be real, leads to a strictly smaller set of densities. In fact,

$$C_q^{\mathbb{R}}(3, 3, 3, 2, 2, 2) \subsetneq C_q(3, 3, 3, 2, 2, 2),$$

and there is a linear functional that is strictly smaller on the real set [RTW⁺21]. Thus, even if one allows for real Hilbert spaces of arbitrarily large dimension, there are densities in the larger set which use finite dimensional complex Hilbert spaces and can not be approximated using finite dimensional real Hilbert spaces of arbitrarily high dimension.

For some time there was debate in the physics community about whether the use of complex Hilbert spaces really were necessary to capture the quantum world. Even Schrödinger considered complex Hilbert spaces as probably unnecessary and referred to their use in quantum mechanics as *unpleasant* [EPK11]. Recently, experiments have been conducted, using the above separating functional, to show that this larger value is attainable [LMW⁺22]. Hence, complex Hilbert spaces are necessary to model quantum phenomena. Briefly, the *real world* is actually complex.

4. SYNCHRONOUS GAMES AND SYNCHRONOUS CORRELATIONS

A game $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ is called **synchronous** provided that $I_A = I_B$, $O_A = O_B$ and

$$V(a, b|x, x) = 0, \forall x \in I_A, a \neq b.$$

This last condition says that whenever Alice and Bob receive the same input, to win they must produce the same output. It should be noted that producing the same output is no guarantee that they win, i.e., it is possible that some outputs are disallowed. Synchronous games were introduced in [HMPS19].

For example, the CHSH game is not a synchronous game. If Alice and Bob both receive 1 for their input, then their outputs, a, b , must satisfy $a + b = 1 \cdot 1$, so to win they must produce different outputs.

Graph coloring games are a good family of synchronous games. These games start with a graph and some fixed number of colors. The inputs are the vertices of the graph and the outputs are colors. Alice and Bob are each given a vertex and must produce colors. If they are given adjacent vertices, they must produce a pair of distinct colors. If they are given the same vertex, they must produce the same color. When they are given a pair of vertices that are neither adjacent or identical, then they win no matter what colors they produce. We will study this family of games later in detail.

To simplify notation for synchronous games we set $I_A = I_B = I$, $O_A = O_B = O$, and $n_A = n_B = n$, $k_A = k_B = k$ and write the game as $\mathcal{G} = (I, O, V)$.

Note that a perfect density for a synchronous game must satisfy

$$p(a, b|x, x) = 0, \forall x, a \neq b.$$

Densities with this property are called *synchronous densities*.

Note that if $f, g : I \rightarrow O$ is a pair of functions that yield a perfect strategy for a synchronous game, then necessarily $f = g$.

Remarkably, this simple constraint on densities, allows us to reduce states on tensor products to traces on a single algebra. The connection between synchronous correlations and traces was first made by S. Severini, D. Stahlke, I.G. Todorov, A. Winter and the second author [PSS⁺16] in their work on graph coloring games. The definition of synchronous games was formally introduced in the work of J.W. Helton, K.P. Meyer, M. Satriano and the second author [HMPS19], partially as a means to explain what was special about the graph coloring games.

We write $C_t^s(n, k)$ for the subset of synchronous densities in $C_t(n, n, k, k)$ for $t = \text{loc}, q, qs, qa, qc$.

The game constructed in [JNV⁺20] with a perfect qc-strategy but no perfect qa-strategy, is actually a synchronous game. We will see that studying synchronous games leads directly to results about traces, and hence, to a negative solution to Connes' Embedding Problem that does not use Kirchberg's equivalence. The connection between synchronous correlations and traces was first made by the S. Severini, D. Stahlke, I.G. Todorov, A. Winter and the second author [PSS⁺16].

Theorem 4.1 ([PSS⁺16]). (1) $p \in C_{qc}^s(n, k)$ if and only if there exists a unital C^* -algebra with a tracial state $(\mathcal{A}, \tau)$ and an (n, k) -PVM $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ in $\mathcal{A}$ such that

$$p(a, b|x, y) = \tau(e_{x,a}e_{y,b}).$$

(2) $p \in C_q(n, k)$ if and only if in the above representation we can assume that $\mathcal{A}$ is finite dimensional.

By the GNS theorem, statement (1) is equivalent to requiring that there is a tracial state τ on $\mathbb{C}(\mathbb{F}(n, k))$ with $p(a, b|x, y) = \tau(e_{x,a}e_{y,b})$.

The remaining cases were described in work of S. Kim, C. Schafhauser and the second author [KPS18].

Theorem 4.2 ([KPS18]). (1) $C_{qs}^s(n, k) = C_q^s(n, k)$, $\forall n, k$,
 (2) $C_{qa}^s(n, k)$ is the closure of the set $C_q^s(n, k)$,
 (3) $p(a, b|x, y) \in C_{qa}^s(n, k)$ if and only if there exists an (n, k) -PVM $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\} \subseteq \mathcal{R}^\omega$ such that

$$p(a, b|x, y) = \tau_\omega(e_{x,a}e_{y,b}).$$

A much clearer proof of (2) is given by T. Vidick [Vid22].

It is important to note that if we start with an (n, k) -POVM $\{p_{x,a}\}$ in a tracial C^* -algebra $(\mathcal{A}, \tau)$ and set $p(a, b|x, y) = \tau(p_{x,a}p_{y,b})$ then this density need not be synchronous, since $p_{x,a}p_{x,b}$ need not be 0.

One consequence of (2) is the following:

Theorem 4.3 ([KPS18, DP16]). *Connes' embedding problem has a positive answer if and only if*

$$C_{qa}^s(n, k) = C_{qc}^s(n, k), \quad \forall n, k.$$

Of course to show that the embedding problem has a negative answer we only need the “only if” part of the above result. A very clean direct proof of this implication is given by A. Frei [Fre22].

4.1. The Fundamental Orthogonality Relations of a Synchronous Game. Let $\mathcal{G} = (I, O, V)$ be an n -input, k -output synchronous game. We say that an (n, k) -PVM $\{e_{x,a} : x \in I, a \in O\}$ in an abstract $*$ -algebra satisfies the **fundamental orthogonality relations**(FOR) of $\mathcal{G}$ provided that:

$$V(a, b, x, y) = 0 \implies e_{x,a}e_{y,b} = 0.$$

Note that the synchronous condition implies $e_{x,a}e_{x,b} = 0, \forall a \neq b$, which while automatically true for an (n, k) -PVM of operators on a Hilbert space is not true for general $*$ -algebras, except for $k \leq 3$. Indeed there exist $*$ -algebras with 4 elements p_1, p_2, p_3, p_4 satisfying $p_i^2 = p_i^* = p_i$ and $p_1 + p_2 + p_3 + p_4 = 1$ and yet $p_i p_j \neq 0$ for $i \neq j$.

To cement these ideas, suppose that we consider the game of trying to k color a graph on n vertices. For this game we seek elements $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ in a $*$ -algebra. The FOR relations are that:

- (1) $e_{x,a} = e_{x,a}^* = e_{xx,a}^2, \forall x, a,$
- (2) $e_{x,a}e_{x,b} = 0, a \neq b,$
- (3) $\sum_{a=1}^k e_{x,a} = 1, \forall x,$
- (4) and whenever (x, y) is an edge in the graph, $e_{x,a}e_{y,a} = 0, .$

Notice that the first three relations are the relations needed to be an (n, k) -PVM, and the 4th relation is determined by the particular graph.

Recall that a tracial state τ is called *faithful* if $\tau(z^*z) = 0 \implies z = 0$. Given a C^* -algebra with a tracial state $(\mathcal{A}, \tau)$ it is readily checked that

$$\mathcal{I} := \{z : \tau(z^*z) = 0\},$$

is a closed 2-sided ideal and that τ defines a faithful tracial state on the quotient C^* -algebra $\mathcal{A}/\mathcal{I}$. Using this fact and the above results one obtains:

Theorem 4.4. *Let $\mathcal{G} = (I, O, V)$ be a synchronous n -input k -output game. Then:*

- (1) $\mathcal{G}$ has a perfect loc-strategy if and only if there exists an (n, k) -PVM in an abelian C^* -algebra satisfying the FOR for $\mathcal{G}$,
- (2) $\mathcal{G}$ has a perfect q -strategy if and only if there exists an (n, k) -PVM in a matrix algebra satisfying the FOR for $\mathcal{G}$,
- (3) $\mathcal{G}$ has a perfect qa -strategy if and only if there exists an (n, k) -PVM in $\mathcal{R}^\omega$ satisfying the FOR for $\mathcal{G}$,
- (4) $\mathcal{G}$ has a perfect qc -strategy if and only if there exists an (n, k) -PVM in a tracial C^* -algebra satisfying the FOR for $\mathcal{G}$.

Since this result is not in the literature, we sketch the proof. Note that if $p(a, b, x, y) = \tau(e_{x,a}e_{y,b})$ is a perfect strategy for the game, then $V(a, b, x, y) = 0 \implies 0 = \tau(e_{x,a}e_{y,b}) = \tau((e_{x,a}e_{y,b})^*(e_{x,a}e_{y,b}))$. So if τ is a faithful trace, then $e_{x,a}e_{y,b} = 0$ and we have that the FOR are met in the quotient algebra $\mathcal{A}/\mathcal{I}$. The proof of (i) follows from observing that if $\mathcal{A}$ is abelian, then $\mathcal{A}/\mathcal{I}$ is abelian. The proof of (ii) from the fact that if $\mathcal{A}$ is finite dimensional, then $\mathcal{A}/\mathcal{I}$ is finite dimensional and so a direct sum of matrix algebras and the FOR's must be met in each matrix algebra. The proof of (iii) follows from observing that given $\pi : \mathcal{A} \rightarrow \mathcal{R}^\omega$, since τ^ω is a faithful trace, the elements $\pi(e_{x,a})$ satisfy the FOR. The proof of (iv) is similar.

The result of [JNV⁺20] now leads to:

Corollary 4.5. *There is an FOR for a synchronous game that can be satisfied in a tracial C*-algebra that is not satisfied by any (n, k) -PVM in $\mathcal{R}^\omega$.*

This is a somewhat stronger negation of the Connes' Embedding Problem. It shows that the set of FOR's that can be satisfied in $\mathcal{R}^\omega$ is a smaller family than the FOR's that can be satisfied in a tracial C*-algebra.

Paddock and Slofstra [PS23] have constructed a synchronous game whose FOR can be satisfied in a C*-algebra but not in a tracial C*-algebra.

We now have synchronous games that separate the FOR's that can be satisfied in abelian C*-algebras, matrix algebras, $\mathcal{R}^\omega$, tracial C*-algebras, and arbitrary C*-algebras.

These results suggest that studying “satisfiability of FOR's” might lead to improved understanding of Connes' Embedding Problem and, via Kirchberg's work, of tensor norms on products of free group algebras.

4.2. The Algebra of a Synchronous Game. The algebra of a synchronous game was introduced in [HMPS19]. It is best thought of as the universal *-algebra for the FOR of the game.

Recall that $\mathbb{C}(\mathbb{F}(n, k))$ has generators $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ that form an (n, k) -PVM with the property that $e_{x,a}e_{x,b} = 0, \forall a \neq b$. Moreover, it is the “universal” *-algebra for an (n, k) -PVM satisfying this extra conditions. By this we mean that if one has a unital *-algebra $\mathcal{A}$ with elements $\{f_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ forming an (n, k) -PVM satisfying $f_{x,a}f_{x,b} = 0, \forall a \neq b$, then there is a unital *-homomorphism $\pi : \mathbb{C}(\mathbb{F}(n, k)) \rightarrow \mathcal{A}$ with $\pi(e_{x,a}) = f_{x,a}, \forall x, a$.

Given a synchronous game $\mathcal{G} = (I, O, V)$ with n -inputs and k -outputs we define the *-algebra of $\mathcal{G}$, denoted $\mathcal{A}(\mathcal{G})$ to be the quotient of $\mathbb{C}(\mathbb{F}(n, k))$ by the 2-sided *-ideal $\mathcal{J}(\mathcal{G})$ generated by

$$\{e_{x,a}e_{y,b} : V(a, b, x, y) = 0\}.$$

Note that because this is a *-ideal whenever $V(a, b, x, y) = 0$ then $e_{y,b}e_{x,a} \in \mathcal{J}(\mathcal{G})$ even though it is not necessary that $V(b, a, y, x) = 0$. The reason that this is desirable is because for perfect strategies

$$V(a, b, x, y) = 0 \implies 0 = p(a, b|x, y) = \tau(e_{x,a}e_{y,b}) = \tau(e_{y,b}e_{x,a}) = p(b, a|y, x).$$

In particular, we have that if $p(a, b|x, y)$ is a perfect strategy for $\mathcal{G} = (I, O, V)$ then it is also a perfect strategy for the game $\widehat{\mathcal{G}} = (I, O, \widehat{V})$ where $\widehat{V}(a, b, x, y) = V(a, b, x, y) \cdot V(b, a, y, x)$. Indeed, some authors prefer to make one of the conditions in the definition of synchronous to be that the “rules” are *symmetric*, meaning that $V(a, b, x, y) = V(b, a, y, x)$.

Note that we have changed notations a bit. It is not the $e_{x,a}$ that satisfy the FOR of the game, but rather their images $\hat{e}_{x,a} = e_{x,a} + \mathcal{J}(\mathcal{G})$ in the quotient.

To fix these ideas, suppose that we consider the game of trying to 4 color the complete graph on 5 vertices. Recall that “complete” means that every pair of vertices is connected. Let us write $\hat{e}_{x,a}$ for the image of the element $e_{x,a}$ in the quotient $\mathcal{A}(\mathcal{G}) = \mathbb{C}(\mathbb{F}(5, 4))/\mathcal{J}(\mathcal{G})$. So the algebra of this game $\mathcal{A}(\mathcal{G})$ is generated by elements $\{\hat{e}_{x,a} : 1 \leq x \leq 5, 1 \leq a \leq 4\}$. The relations are that:

- (1) $\hat{e}_{x,a} = \hat{e}_{x,a}^* = \hat{e}_{xx,a}^2, \forall x, a,$
- (2) $\hat{e}_{x,a}\hat{e}_{x,b} = 0, a \neq b,$
- (3) $\sum_{a=1}^4 \hat{e}_{x,a} = 1, \forall x,$
- (4) $\hat{e}_{x,a}\hat{e}_{y,a} = 0, x \neq y.$

If we set $p_a = \sum_{x=1}^5 \hat{e}_{x,a}$ then $p_a = p_a^*$ and

$$p_a^2 = \sum_{x,y=1}^5 \hat{e}_{x,a}\hat{e}_{y,a} = \sum_{x=1}^5 \hat{e}_{xx,a}^2 = p_a.$$

So each p_a is a self-adjoint idempotent in the $*$ -algebra. However,

$$\sum_{a=1}^4 p_a = \sum_{x=1}^5 \sum_{a=1}^4 \hat{e}_{x,a} = 5 \cdot 1.$$

So this algebra contains 4 “projections” that sum to 5 times the identity. From this it is clear that this algebra, if it is non-zero, i.e., $\mathcal{J}(\mathcal{G}) \neq \mathbb{C}(\mathbb{F}(5, 4))$, cannot be represented as operators on a Hilbert space. Nevertheless, in [HMPS19] it is proven that $\mathcal{J}(\mathcal{G}) \neq \mathbb{C}(\mathbb{F}(5, 4))$. So this gives an example of a (non-zero) game algebra that cannot be represented as operators on a Hilbert space.

The utility of the game algebra is that determining if games have perfect strategies becomes a question about representations of the algebra. Restating the above results we have the following.

Theorem 4.6 ([HMPS19]). *Let $\mathcal{G}$ be a synchronous game. Then*

- (1) $\mathcal{G}$ has a perfect deterministic strategy if and only if $\mathcal{A}(\mathcal{G})$ has a non-zero multiplicative linear functional,
- (2) $\mathcal{G}$ has a perfect q -strategy if and only if there is a unital $*$ -homomorphism of $\mathcal{A}(\mathcal{G})$ into some matrix algebra,
- (3) $\mathcal{G}$ has a perfect qa -strategy if and only if there is a unital $*$ -homomorphism of $\mathcal{A}(\mathcal{G})$ into $\mathcal{R}^\omega$,

- (4) $\mathcal{G}$ has a perfect qc-strategy if and only if there is a unital $*$ -homomorphism of $\mathcal{A}(\mathcal{G})$ into some unital C^* -algebra with a tracial state.

In [HMPS19] examples were given of synchronous games for which $\mathcal{J}(\mathcal{G}) = \mathbb{C}(\mathbb{F}(n, k))$ and so $\mathcal{A}(\mathcal{G}) = 0$ and, hence, can have no perfect strategies of any of these types. The example above is of a game for which $\mathcal{A}(\mathcal{G}) \neq 0$ and yet this unital $*$ -algebra had no unital $*$ -homomorphisms into the bounded operators on any Hilbert space. This will be discussed more fully in the section on graph coloring. Paddock and Slofstra [PS23] gave the first example for which $\mathcal{A}(\mathcal{G})$ had a unital $*$ -homomorphism into a unital C^* -algebra but $\mathcal{G}$ had no perfect qc-strategies and hence it could never have a unital $*$ -homomorphism into a unital C^* -algebra with a trace.

Bene Watts, Helton, and Klepp also give a nice purely algebraic characterization in terms of $*$ -algebras for when a synchronous game has a perfect qc-strategy.

Theorem 4.7 ([BHK23]). *Let $\mathcal{G} = (I, O, V)$ be a synchronous game with n -inputs and k -outputs, let*

$$\mathcal{F} = \{e_{x,a}e_{y,b} : V(a, b, x, y) = 0 \text{ or } V(b, a, y, x) = 0\},$$

*let $\mathcal{L} \subseteq \mathbb{C}(\mathbb{F}(n, k))$ be the left ideal generated by $\mathcal{F}$, let $\mathcal{R} \subseteq \mathbb{C}(\mathbb{F}(n, k))$ be the right ideal generated by $\mathcal{F}$, let $\mathcal{C} = \text{span}\{zw - wz : z, w \in \mathbb{C}(\mathbb{F}(n, k))\}$ denote the linear span of all commutators, and let SOS denote the cone generated by all $z^*z, z \in \mathbb{C}(\mathbb{F}(n, k))$, i.e., the positive cone of the $*$ -algebra. Then $\mathcal{G}$ has a perfect qc-strategy if and only if*

$$-1 \notin \mathcal{L} + \mathcal{R} + SOS + \mathcal{C}.$$

This beautiful result reduces the problem of determining if there exists a perfect qc-density to a purely symbolic calculation without the need to actually produce a density or a representation into a tracial C^* -algebra. It also shows the asymmetry between the problem of checking a proof that a density does not exist, versus checking a proof that a density does exist.

4.3. Synchronous Values of Games. For a synchronous game $\mathcal{G} = (I, O, V)$ with n inputs and k outputs and density π it is natural to study *synchronous values* of the game. By this we mean for $t = loc, q, qa, qsqc, ns$, the quantities,

$$\omega_t^s(\mathcal{G}, \pi) := \sup\left\{\sum_{a,b,x,y} \pi(x, y)V(a, b, x, y)p(a, b|x, y) : p \in C_t^s(n, k)\right\}.$$

By the results of [KPS18],

$$\omega_q^s(\mathcal{G}, \pi) = \omega_{qs}^s(\mathcal{G}, \pi) = \omega_{qa}^s(\mathcal{G}, \pi).$$

Synchronous values of games were introduced in [DP16] and are studied extensively in [HMN⁺24].

The value $\omega_{loc}^s(\mathcal{G}, \pi)$ is also obtained by taking the supremum over all deterministic strategies $f, g : I \rightarrow O$ such that $f = g$. For this reason,

some quantities, such as the Maxcut of a graph, are shown to be related to the synchronous deterministic values of a game as opposed to the ordinary value. This viewpoint lends itself more readily to quantum analogues of such quantities, by considering the corresponding synchronous quantum value of the same game.

For example, [HMN⁺24] uses this viewpoint to develop a quantum version of the Maxcut of a graph, which they prove is the quantum synchronous value of a game, and using results of Tsirelson show that this quantum Maxcut is solvable by an SDP, as opposed to the classical Maxcut which is an NP-hard problem.

If one sets

$$R = \sum_{a,b,x,y} \pi(x,y) V(a,b,x,y) e_{x,a} e_{y,b} e_{x,a} \in C^*(\mathbb{F}(n,k)),$$

then [HMN⁺24] proves that

$$\omega_{qc}^s(\mathcal{G}, \pi) = \sup\{\tau(R) : \tau \text{ a tracial state on } C^*(\mathbb{F}(n,k))\} = \inf\{\|R - W\| : W \in \mathcal{C}\},$$

where $\mathcal{C}$ is the span of the commutators and the norm is in $C^*(\mathbb{F}(n,k))$.

4.4. Multipartite Theory of Synchronous Densities. There is very little literature on synchronous games for three or more players. One reason is the following. Suppose that we have a three person game with common input set $I_A = I_B = I_C$ with n elements and a common output set $O_A = O_B = O_C$ with k elements. We could define “synchronous” to mean that whenever any pair of players receive the same input, that pair must give the same output, or a somewhat weaker condition is that when all three players receive the same input, then they must all three give the same output. A perfect density $p(a,b,c|x,y,z) \in C_{qc}(n,n,n,k,k,k)$ for this weaker version of synchronous satisfies

$$p(a,b,c|x,x,x) = 0 \text{ unless } a = b = c.$$

It turns out that any such density p is a local density. Because this result is not in the literature, we sketch a proof here.

Suppose that one has three (n,k) -PVM's $\{E_{x,a}\}$, $\{F_{y,b}\}$, $\{G_{z,c}\}$ on a Hilbert space $\mathcal{H}$ such that

$$E_{x,a} F_{y,b} = F_{y,b} E_{x,a}, E_{x,a} G_{z,c} = G_{z,c} E_{x,a}, F_{y,b} G_{z,c} = G_{z,c} F_{y,b},$$

and a unit vector ϕ such that

$$p(a,b,c|x,y,z) = \langle \phi | E_{x,a} F_{y,b} G_{z,c} \phi \rangle,$$

that satisfies the above “synchronous” condition. Without loss of generality we can assume that $\mathcal{H}$ is the cyclic subspace generated for the algebra generated by the three PVM's applied to ϕ . Using the Schwarz inequality and the fact that

$$1 = \sum_a p(a,a,a|x,x,x) = \sum_a \langle \phi | E_{x,a} F_{x,a} G_{x,a} \phi \rangle,$$

one shows that $E_{x,a}\phi = F_{x,a}\phi = G_{x,a}\phi$. From this one shows that $\mathcal{H}$ is the cyclic subspace for the algebra generated by $\{G_{z,c}\}$ applied to ϕ and that consequently, $E_{x,a} = F_{x,a}, \forall x, a$. Hence, $E_{x,a}E_{y,b} = E_{y,b}E_{x,a}$ and so we have a local density.

A similar result holds for more than three players. Thus, $C_{loc}^s = C_q^s = C_{qa}^s = C_{qc}^s$ as soon as we are tripartite or more.

However, the synchronous nonsignalling densities are still larger than the local densities even in the multipartite case and there is some interest in studying values of synchronous ns-strategies for multipartite games. See [BHTT23b, LMP⁺20, TT24].

4.5. Beyond Synchronous. There have been various families of games introduced with properties that reflect some of the properties of synchronous games—in particular, the aptly named *mirror games* and *imitation games*. See [BHTT23b, LMP⁺20].

We have a nice general theory of games but currently we have only presented one game. In the following sections we turn our attention to the many families of games that have been studied extensively.

5. XOR GAMES

A game $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ is called an **XOR game** provided that $O_A = O_B = \mathbb{Z}_2$, the binary field and there is a function

$$f : I_A \times I_B \rightarrow \mathbb{Z}_2,$$

such that

$$V(a, b, x, y) = 1 \iff a + b = f(x, y),$$

with arithmetic in $\mathbb{Z}_2$. For example, the CHSH game is the XOR game with $I_A = I_B = \mathbb{Z}_2$ and $f(x, y) = xy$.

This family of games were introduced by Cleve, Hoyer, Toner, and Watrous [CHTW04], who were the first to study nonlocal strategies for games. They also realized that the inequality introduced by [CHSH69] to separate quantum densities from local densities could be interpreted as the q-value of a game, which they dubbed the CHSH game.

It is not difficult to show that given an XOR game $\mathcal{G}$ with density π that the various values of the game are all attained by densities that satisfy

$$p_A(0|x) = p_A(1|x) = p_B(0|y) = p_B(1|y) = 1/2.$$

Moreover, Tsirelson's work [Tsi93] implies that if such a density $p(a, b|x, y) \in C_{qc}$, then $p(a, b|x, y) \in C_q$. For this particular interpretation of Tsirelson's work see the online lecture notes [Pau16]. Hence, for XOR games one has $\omega_q(\mathcal{G}, \pi) = \omega_{qc}(\mathcal{G}, \pi)$.

Another remarkable result from [CHTW04] is that the quantum value of an XOR game is given by an SDP. In particular, they prove the following.

Let $|I_A| = n$, $|I_B| = m$ and define $A_{\mathcal{G}, \pi} \in M_{n, m}$ by

$$A_{\mathcal{G}, \pi} = ((-1)^{f(x, y)})\pi(x, y),$$

and

$$B_{\mathcal{G},\pi} = \begin{pmatrix} 0 & A_{\mathcal{G},\pi} \\ A_{\mathcal{G},\pi}^T & 0 \end{pmatrix} \in M_{n+m,n+m}.$$

Consider the elliptope,

$$\mathcal{E}_{n+m} = \{P \in M_{n+m} : \text{diag}(P) = I_{n+m} \text{ and } P \text{ is positive semidefinite}\}.$$

Then the results of [CHTW04] imply that

$$\omega_q(\mathcal{G}, \pi) = \omega_{qc}(\mathcal{G}, \pi) = 1/2 + 1/2 \sup\{\text{Tr}(B_{\mathcal{G},\pi}P) : P \in \mathcal{E}_{n+m}\}.$$

For a proof of this restatement of [CHTW04] see [HMN⁺24, Theorem 4.7].

An XOR game is synchronous provided $I_A = I_B$ and the defining function f satisfies $f(x, x) = 0, \forall x$. In [HMN⁺24] it is shown that for such games

$$\omega_q^s(\mathcal{G}, \pi) = \omega_{qc}^s(\mathcal{G}, \pi) = 1/2 + 1/2 \sup\{\text{Tr}((A_{\mathcal{G},\pi} + A_{\mathcal{G},\pi}^T)P) : P \in \mathcal{E}_n\},$$

and this result is used to give examples of synchronous games with $\omega_q^s(\mathcal{G}, \pi) < \omega_q(\mathcal{G}, \pi)$. Thus for synchronous games, the optimal strategy might not even be synchronous.

6. LINEAR CONSTRAINT SYSTEM GAMES

These games had an enormous influence on the development of the theory of nonlocal games.

Recall that if p is a prime number then the set of integers modulo p , $\mathbb{Z}_p$ equipped with addition modulo p and multiplication modulo p is a *field* denoted $\mathbb{Z}_p$. The most familiar of these is the *binary field* $\mathbb{Z}_2 = \{0, 1\}$, where $1 + 1 \equiv 0$.

Suppose that we are given a system of n linear equations in m variables over $\mathbb{Z}_p$:

$$\sum_{j=1}^m a_{i,j}x_j = b_i, 1 \leq i \leq n,$$

or in matrix vector notation,

$$A\vec{x} = \vec{b}.$$

There are two versions of the **linear constraint system game**. The original version is not synchronous and later a synchronous game was introduced that has since been shown to be equivalent to the original game, in a sense that we will make precise below.

First we discuss the non-synchronous game. In this game Alice is given an equation, i.e., $I_A = \{1, \dots, n\}$ and Bob is given a variable, $I_B = \{1, \dots, m\}$. Alice must return values for each of the variables in her equation and Bob must give a value to his variable. They win if Alice's variable values satisfy the equation that she was given and if the value that Bob assigned to his particular variable is the same as the value that Alice gave to that variable. If Bob's variable is not one of the variables that appears in Alice's equation, then they automatically win.

Note that in this description of the game, Bob's output set is $O_B = \mathbb{Z}_p$ while Alice's output set is a union of vector spaces over $\mathbb{Z}_p$ with the dimensions depending on the number of non-zero entries in her row. An alternate description that is a bit fussier to state, but in the end tidier, is to demand that when Alice receives the i -th equation, she returns a vector in $\mathbb{Z}_p^m$ that satisfies her equation and is zero for each j such that $a_{i,j} = 0$. As before, for the players to win, Bob's variable value must be equal to the corresponding entry in Alice's vector, except when $a_{i,j} = 0$, in which case the only winning requirement is that Alice's vector satisfy the equation and is 0 in the appropriate entries.

We shall denote this game by $LCS(A, \vec{b})$, with p understood. In the binary case, i.e., $p = 2$, this game is sometimes denoted $BCS(A, \vec{b})$.

It is not difficult to show that this game has a perfect deterministic strategy if and only if the system of equations has a solution.

The most famous and earliest studied of these games is *Mermin's magic square*, which is the system of 6 equations in 9 variables over the binary field. If we imagine the variables as arranged into a 3×3 array, then the 6 equations are that each row sums to 0, while each column sums to 1. If we first sum all rows, then we see that the sum of all variables must be 0, while if we sum all columns, then we see that the sum of all variables must be 1. From this it follows that this game has no perfect deterministic strategy and so no perfect local strategy. However, it does have a perfect q-strategy [Mer90, CM14, CLS17].

Thus, in some quantum sense these equations have a solution over the binary field. There are still open questions about Mermin's Magic Square game over $\mathbb{Z}_p$ for primes other than 2. For example, it is not known if these games have perfect qc-strategies if and only if they have perfect q-strategies.

Slofstra [Slo19a, Slo19b] proved that C_q and C_{qs} are not closed using linear constraint system games.

For this game there is a group $\Gamma(A, \vec{b})$, called the *solution group* of the game, defined by generators and relations. The representation theory of this group determines whether or not this game has perfect strategies.

The solution group has $m + 1$ generators, denoted $U_1, \dots, U_m, J$ and relations given as follows:

- $U_1^p = \dots = U_m^p = J^p = 1$, where 1 denotes the group identity,
- for each i ,

$$U_1^{a_{i,1}} \dots U_m^{a_{i,m}} = J^{b_i},$$

- if for some i , $a_{i,j}a_{i,k} \neq 0$, then $U_j U_k = U_k U_j$,
- $U_j J = J U_j$ for all j .

The role of the solution group is developed in [CM14, CLS17, CS17, Slo19a, Slo19b]. In particular, [CS17] extended the theory from the case $p = 2$ to general primes. For example, $LCS(A, \vec{b})$ has a perfect q-strategy if and only if there is a unitary representation π of $\Gamma(A, \vec{b})$ on a finite dimensional Hilbert space with the property that $\pi(J) = \omega I$ where $\omega = e^{2\pi i/p}$.

The synchronous game $\text{syncLCS}(A, \vec{b})$ is defined as follows.

Alice and Bob's inputs are $I = \{1, \dots, n\}$, i.e., the label of an equation, and their outputs are $O = \mathbb{Z}_p^m$ vectors. If Alice receives i_1 and Bob receives i_2 and they return vectors $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_m)$ then they win if and only if:

- $\sum_{j=1}^m a_{i_1,j} v_j = b_{i_1}$ and $\sum_{j=1}^m a_{i_2,j} w_j = b_{i_2}$,
- $a_{i_1,j} = 0 \implies v_j = 0$ and $a_{i_2,j} = 0 \implies w_j = 0$,
- $a_{i_1,j} a_{i_2,j} \neq 0 \implies v_j = w_j$.

Notice that these rules do imply that if $i_1 = i_2$, i.e., they both receive the same input, then $v = w$, so that the game is synchronous.

This synchronous game was first introduced in [KPS18] for the case $p = 2$ and extended to general primes in [Gol21].

Note that if we let $\langle J - \omega 1 \rangle \subseteq \mathbb{C}(G)$ denote the 2-sided ideal in the $*$ -algebra of the group generated by the single element $J - \omega 1$, then π is a unitary representation of $\Gamma(A, \vec{b})$ satisfying $\pi(J) = \omega I$ if and only if the extension of π to the group algebra vanishes on this ideal. Thus, the condition $\pi(J) = \omega I$ on a unitary representation π of $\Gamma(A, \vec{b})$ needed for the game $\text{LCS}(A, \vec{b})$ to have perfect strategies, corresponds precisely to the unital $*$ -homomorphisms of the $*$ -algebra quotient $\mathbb{C}(\Gamma(A, \vec{b})/\langle J - \omega 1 \rangle)$.

A. Goldberg [Gol21, Theorem 4.1] proves that the algebra of the synchronous game $\text{syncLCS}(A, \vec{b})$ is unittally $*$ -isomorphic to $\mathbb{C}(\Gamma(A, \vec{b})/\langle J - \omega 1 \rangle)$. Consequently, the game $\text{LCS}(A, \vec{b})$ has a perfect t-strategy if and only if $\text{syncLCS}(A, \vec{b})$ has a perfect t-strategy. The earlier results about perfect t-strategies for $\text{LCS}(A, \vec{b})$ can be derived from this fact and the general theorem about conditions for perfect strategies for synchronous games. In the paper [KPS18] some of these equivalences for perfect strategies of the two versions of the game had been shown for the binary case.

Goldberg's result was one of the first results to really use the algebra of a synchronous game to clarify and extend other results. For a further study of $*$ -isomorphism of game algebras see [Har22].

It is still unknown if these games can have perfect qc-strategies without perfect qa-strategies, i.e., if the separation of C_{qa} and C_{qc} can be achieved with a linear systems game.

6.1. Linear equations over $\mathbb{Z}$. Given a system of equations over $\mathbb{Z}$, $A\vec{x} = \vec{b}$ and a bound $-N \leq x_j \leq +N, \forall j$ (so that the output sets are bounded), one can consider an equation-variable game and a synchronous equation-equation game in parallel with the above constructions. Very little is known, including if the two versions of the game are "equivalent" in some appropriate sense and how these games are related as N varies.

6.2. Other Systems of Equations. Given n functions, $f_i : \mathbb{Z}_p^m \rightarrow \mathbb{Z}_p, 1 \leq i \leq n$ and $b_1, \dots, b_m \in \mathbb{Z}_p$ one can study games for solving the system of

equations, $f_i(x_1, \dots, x_m) = b_i$, $1 \leq i \leq n$, as above there is an equation-variable and a synchronous equation-equation game that can be introduced. For the most complete work in this direction see the paper of Paddock and Slofstra [PS23].

Satisfiability of Boolean equations is directly related to logic and the fact that more equations are “quantum satisfiable” is related to different logics. For some work in this direction see [VW15, GH23]. There is also a huge literature in computer science on various quantum notions of hardness of proof and this played a central role in the negation of the CEP [JNV⁺20].

7. GAMES BASED ON GRAPHS

By a **graph** we mean a pair $G = (V, E)$ where V is a, generally, finite set called the *vertices* and $E \subset V \times V$ is a subset called the *edges* satisfying:

- (loopless) $(x, x) \notin E$ for any $x \in V$,
- (undirected) $(x, y) \in E \implies (y, x) \in E$.

We assume that the reader has some familiarity with concepts from graph theory such as *colorings*, *cliques*, *independent sets* and the corresponding notions of the *chromatic number*, *clique number*, and *independence number*, of a graph.

In this section we discuss synchronous games for these concepts. All of these quantities can be determined by *graph homomorphisms*, so we begin there.

7.1. The Graph Homomorphism Game. Given two graphs $G_i = (V_i, E_i)$, $i = 1, 2$ a *homomorphism* from G_1 to G_2 is a function $f : V_1 \rightarrow V_2$ with the property that if $(x, y) \in E_1$ then $(f(x), f(y)) \in E_2$. We write $G_1 \rightarrow G_2$ to indicate that there exists a graph homomorphism from G_1 to G_2 .

Graph homomorphisms are convenient for capturing many of the parameters studied in graph theory. For instance, if K_k denotes the *complete graph* on k vertices, i.e., every unequal pair is an edge, then it is not hard to see that G has a k -colouring if and only if $G \rightarrow K_k$. Thus, the *chromatic number* $\chi(G)$ is the smallest k for which $G \rightarrow K_k$.

Similarly, a *clique* in G is a subset of vertices such that every pair is connected by an edge. It is not hard to see that G has a clique of size k if and only if $K_k \rightarrow G$ and the *clique number*, $\omega(G)$, is the largest such k .

A set of vertices in G is called *independent* if they contain no edges. Let G^c denote the *complement of G* , i.e., the graph with the same vertices, but with (x, y) an edge in G^c if and only if $x \neq y$ and $(x, y) \notin E$. Thus, G has an independent set of size k if and only if $K_k \rightarrow G^c$ and the *independence number*, $\alpha(G)$, is the largest such k .

The **graph homomorphism game**, $\text{Hom}(G_1, G_2)$, is the synchronous game with input set V_1 and output set V_2 and losing tuples (a, b, x, y) given by the set

$$\{(a, b, x, y) : (x, y) \in E_1, (a, b) \notin E_2\} \cup \{(a, b, x, x) : x \in V_1, a \neq b\}.$$

Note that if $x \neq y$ and (x, y) is not an edge, then (a, b, x, y) is a winning tuple for all a, b .

It is not hard to see that a perfect deterministic strategy for this game is an actual graph homomorphism.

For $t = loc, q, qa, qc, ns$, we write $G_1 \xrightarrow{t} G_2$ to indicate that there is a perfect density for $Hom(G_1, G_2)$ in C_t . In particular,

$$G_1 \rightarrow G_2 \iff G_1 \xrightarrow{loc} G_2.$$

These so-called *quantum graph homomorphisms* are known to be preserved under compositions, i.e., given graphs $G_i = (V_i, E_i)$ on n_i vertices for $i = 1, 2, 3$ we have

$$G_1 \xrightarrow{t} G_2 \text{ and } G_2 \xrightarrow{t} G_3 \implies G_1 \xrightarrow{t} G_3.$$

This follows from two facts. First if $p(a, b|x, y) \in C_t^s(n_1, n_2)$ and $q(c, d|a, b) \in C_t^s(n_2, n_3)$ then

$$r(c, d|x, y) := \sum_{a, b} q(c, d|a, b)p(a, b|x, y) \in C_t^s(n_1, n_3).$$

Moreover, if p is a perfect t -strategy for $Hom(G_1, G_2)$ and q is a perfect t -strategy for $Hom(G_2, G_3)$, then r is a perfect t -strategy for $Hom(G_1, G_3)$.

For $t = loc, q, qa, qc, ns$, the corresponding *quantum chromatic numbers*, $\chi_t(G)$, are defined to be the smallest k for which $G \xrightarrow{t} K_k$. There are also similar definitions via quantum graph homomorphisms for the *quantum clique number*, $\omega_t(G)$, and *quantum independence number*, $\alpha_t(G)$. For an introduction to this literature see [CNM⁺07, MR12, MRV15, MR16a] and the references therein.

Borrowing from above we call $Hom(G, K_k)$ the *k-coloring game*, $Hom(K_k, G)$ the *k-clique game*, and $Hom(K_k, G^c)$ the *k-independent set game*.

The game $Hom(G, K_k)$ is the graph coloring game that we discussed in earlier sections.

In [MR12] it is shown that $G \xrightarrow{ns} K_2$ and hence, $\chi_{ns}(G) \leq 2$ for all graphs.

There is a fairly extensive literature on each of these concepts, often seeking quantum versions of classical results from graph theory.

There are examples of graphs known for which $\alpha_q(G) \neq \alpha_{qa}(G)$ [KPS18] giving yet more examples of games separating C_q from C_{qa} .

Given a graph $G = (V, E)$, if one lets G_∞ denote the one-point suspension of G , i.e., $G_\infty = (V \cup \infty, E_\infty)$ where

- $(x, y) \in E \implies (x, y) \in E_\infty$,
- $x \in V \implies (x, \infty), (\infty, x) \in E_\infty$,

then it is clear that $\chi(G_\infty) = \chi(G) + 1$. Mančinska and Roberson [MR16b] construct a graph with $\chi_q(G_\infty) = \chi_q(G)$. In particular, $\chi_q(G_\infty) < \chi(G_\infty)$, and this is the smallest known example of a graph separating χ and χ_q .

7.2. Universality of the 3-coloring Game. The first author [Har24] proves that the graph 3-coloring game is in a certain sense universal for all synchronous games, strengthening previous work of Ji [Ji13] for quantum finite-dimensional strategies. We make this precise below. Define two synchronous games $\mathcal{G}_1$ and $\mathcal{G}_2$ to be *weakly *-equivalent* if and only if there exist unital *-homomorphisms from $\mathcal{A}(G_1)$ to $\mathcal{A}(G_2)$ and from $\mathcal{A}(G_2)$ to $\mathcal{A}(G_1)$. This can be easily seen to be equivalent to [Har24, Definition 2.2]. It is easy to see that weak *-equivalence is an equivalence relation on the set of synchronous games.

In [Har24, Proposition 2.3] it is shown that if two synchronous games, $\mathcal{G}_1$ and $\mathcal{G}_2$ are weakly *-equivalent, then for $t = loc, q, qa, qc$,

$$\mathcal{G}_1 \text{ has a perfect } t\text{-strategy} \iff \mathcal{G}_2 \text{ has a perfect } t\text{-strategy}.$$

The first author proves [Har24, Theorem 4.5] that each synchronous game is weakly *-equivalent to a graph 3-coloring game. In fact, given a synchronous game $\mathcal{G}$ an explicit graph G is constructed so that $\mathcal{G}$ and $Hom(G, K_3)$ are weak *-equivalent. The independence number and clique numbers enjoy similar universality results (see [MRV15] for the quantum finite-dimensional strategy case, and [Har24] for the general case).

Applying this result to Slofstra's undecidability result, shows that there is no algorithm for determining the graphs satisfying $\chi_q(G) \leq 3$.

Using the results from [JNV⁺20], this result shows that there exists graphs for which $Hom(G, K_3)$ has a perfect qc-strategy but no perfect qa-strategy. This shows that the fundamental orthogonality relations for 3-colorings are enough to separate orthogonality relations satisfiable in a general tracial C*-algebra from those that can be satisfied in $\mathcal{R}^\omega$. But as yet these graphs are only implicitly defined via the results of [JNV⁺20] and no direct construction of such graphs yet exists.

Applying this result to Slofstra's example of a linear system game with a perfect qa-strategy but no perfect q-strategy, yields the existence of a graph G for which $\chi_{qa}(G) \leq 3 < \chi_q(G)$. Earlier, [KPS18] had used Slofstra's result to show that there exist graphs for which $\chi_{qa}(G) < \chi_q(G)$ and $\alpha_q(G) < \alpha_{qa}(G)$.

Because of this universality of quantum chromatic numbers, we look at this area a bit more.

7.3. Hadamard Graphs. Not only can $\chi_q(G) < \chi(G)$, but the family of *Hadamard graphs* are known to have quantum chromatic numbers that are exponentially smaller than their chromatic numbers.

The Hadamard graphs Ω_N are defined as follows:

- the vertex set is all N -tuples of ± 1 , so that Ω_N has 2^N vertices,
- two vertices $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ are adjacent if and only if

$$x \cdot y := \sum_{i=1}^N x_i y_i = 0.$$

Note that if N is odd then $x \cdot y \neq 0$ so the only interesting case is for N even.

It is known that $\chi(\Omega_N) > (1.06)^N$ [FR87], but exact values of the chromatic number are only known for a few values of N . On the other hand $\chi_q(\Omega_N) = N$, for all even N [AHKS06].

There are also complex generalizations of the Hadamard graphs.

If we consider the graph G with uncountably many vertices,

$$V = \mathbb{T}^N := \{\vec{\lambda} = (\lambda_0, \dots, \lambda_{N-1}) : \lambda_i \in \mathbb{C}, |\lambda_i| = 1\} \subseteq \mathbb{C}^N,$$

and $(\vec{\lambda}, \vec{\mu}) \in E \iff \langle \vec{\lambda} | \vec{\mu} \rangle := \sum_{i=1}^N \bar{\lambda}_i \mu_i = 0$, then results of [PT15] show that this graph also has $\chi_q(G) = N$. Very little is known about the chromatic number of these graphs.

Since this result is not explicitly in [PT15], we present a proof here. It also gives a different proof of the inequality $\chi_q(\Omega_N) \leq N$ than is in the literature.

Theorem 7.1 ([PT15]). *Let $V \subseteq \mathbb{C}^N$ with $|V| = n$, where each $v \in V$ is such that $v = (v(0), \dots, v(N-1)) \in \mathbb{C}^N$ with $|v(j)| = 1$ for all j . Define $E = \{(v, w) : v, w \in V, v \perp w\}$ and let $G = (V, E)$. Then $\chi_q(G) \leq N$ (and hence $\chi_q(\Omega_N) \leq N$).*

Proof. Let $\omega = e^{\frac{2\pi i}{N}}$. For each $v \in V$, let $D_v = \text{diag}(v(0), \dots, v(N-1))$ which is unitary. For each $0 \leq k \leq N-1$, let $R_k = \frac{1}{N}(\omega^{(\ell-j)k})_{\ell,j=0}^{N-1} \in M_N$. Set $h_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \omega^{\ell k} e_\ell$, where $\{e_\ell : 0 \leq \ell \leq N-1\}$ is the canonical orthonormal basis for $\mathbb{C}^N$, so that $R_k = h_k h_k^* \geq 0$. This also shows that R_k is a rank one projection. Recall that $\sum_{k=0}^{N-1} (\omega^\ell)^k = 0$ for all $1 \leq \ell \leq N-1$ (it is equal to N when $\ell = 0, N$). Hence,

$$\sum_{k=0}^{N-1} R_k = \frac{1}{N} \left(\sum_{k=0}^{N-1} (\omega^{\ell-j})^k \right) = I_N.$$

Thus these are a set of orthogonal rank one projections.

Let $P_{v,k} = D_v^* R_k D_v$, so that $P_{v,k} \geq 0$ and $\sum_{k=0}^{N-1} P_{v,k} = I$. Then $\{P_{v,k} : 1 \leq v \leq n, 0 \leq k \leq N-1\}$ is an (n, N) -PVM. Hence

$$p(k, j | v, w) = \text{tr}_N(P_{v,k} P_{w,j}) = \text{tr}_N(D_v^* h_k h_k^* D_v D_w^* h_j h_j^* D_w) = |\langle D_v^* h_k | D_w^* h_j \rangle|^2,$$

is in $C_q(n, N)$.

If (v, w) is an edge, then $p(k, k | v, w) = 0$ so that it is a perfect density for $\text{Hom}(G, K_N)$. $\square$

7.4. Fractional Chromatic Numbers. The *fractional chromatic number* of a graph $G = (V, E)$, denoted $\chi_f(G)$, is defined as follows. Given a natural number k , we seek a finite set F and subsets S_v , $v \in V$ of k elements such that

$$(v, w) \in E \implies S_v \cap S_w = \emptyset.$$

Then

$$\chi_f(G) = \inf \left\{ \frac{|F|}{k} \right\},$$

where the infimum is over all such pairs k and F . This value gives a lower bound on the chromatic number, since the chromatic number is the case $k = 1$. Remarkably, $\chi_f(G)$ is always rational and it is computable by a linear program.

Mančinska and Roberson [Rob13, MR16a] were the first to introduce and study quantum analogues of the fractional chromatic number, which they called the *projective rank*, and denoted $\xi_f(G)$. Later Severini, Stahlke, Todorov, Winter and the second author [PSS⁺16] would introduce further quantum analogues in terms of traces and give a game theoretic characterization of these values. We summarize some of these results here, adopting some slightly different notations.

Given a graph $G = (V, E)$ and a tracial C*-algebra $(\mathcal{A}, \tau)$ we seek sets of projections $\{P_v : v \in V\}$ such that:

- $\tau(P_v) = \tau(P_w)$, $\forall v, w \in V$,
- $(v, w) \in E \implies P_v P_w = 0$.

We set $\beta(G, \mathcal{A}, \tau)$ equal to the supremum of this common value of $\tau(P_v)$ over all such families in $\mathcal{A}$. Then the various version of the fractional quantum chromatic numbers are defined as follows:

- $\xi_{loc}(G)^{-1} = \sup \{ \beta(G, \mathcal{A}, \tau) : \mathcal{A} \text{ is abelian} \},$
- $\xi_q(G)^{-1} = \sup \{ \beta(G, \mathcal{A}, \tau) : \mathcal{A} \text{ is finite dimensional} \},$
- $\xi_{qa}(G)^{-1} = \beta(G, \mathcal{R}^\omega, \tau_\omega),$
- $\xi_{qc}(G)^{-1} = \sup \{ \beta(G, \mathcal{A}, \tau) : \mathcal{A} \text{ is a tracial C*-algebra} \}.$

It is not hard to see that

$$\xi_q(G)^{-1} = \sup \{ \beta(G, M_n, tr_n) : n \in \mathbb{N} \},$$

and this right hand side is the quantity $\xi_f(G)$, introduced by Mancinska and Roberson [MR16a]. In [PSS⁺16] $\xi_{qc}(G)$ is denoted $\xi_{tr}(G)$ and it is shown that $\xi_{loc}(G)$ is equal to the classical fractional chromatic number $\chi_f(G)$. Also from the result of [KPS18] that the closure of C_q^s is C_{qa}^s it follows that $\xi_q(G) = \xi_{qa}(G)$.

The results of [MR16a] and [PSS⁺16] show that for $t = loc, q, qa, qc$,

$$\xi_t(G) \leq \chi_t(G).$$

Currently, it is not known if these nonlocal quantum versions are also necessarily rational. In [DP16] it is remarked that if $\xi_q(G)$ is irrational for some graph then that would give another proof that C_q is not closed.

In [PSS⁺16], SDP's are given for computing $\chi_{qc}(G)$ and $\xi_{qc}(G)$.

The game theoretic approach of [PSS⁺16] also gives a way to define a value for $\xi_{ns}(G)$, but since the result of [MR12] shows that $\chi_{ns}(G) \leq 2$, this value is probably uninteresting.

Even though there is no algorithm for deciding if $\chi_q(G) \leq 3$ it is unknown if the problem of determining if $\xi_q(G) \leq 3$ is decidable.

7.5. The Algebra of the Graph Coloring Game. The algebra $\mathcal{A}(\text{Hom}(G, K_k))$ for the game of k -coloring a graph $G = (V, E)$ with n vertices has generators $\{e_{v,a} : 1 \leq v \leq n, 1 \leq a \leq k\}$ that satisfy the axioms to be an (n, k) -PVM, including $e_{v,a}e_{v,b} = 0, \forall a \neq b$ and

$$(v, w) \in E \implies e_{v,a}e_{w,a} = 0, \forall a.$$

These algebras can be quite bizarre and unlike the quotients in [BHK23], these algebras can be nonzero even when there are no perfect qc-strategies.

For example, it is known that $k = \chi(K_k) = \chi_q(K_k) = \chi_{qa}(K_k) = \chi_{qc}(K_k)$. But if we consider the game for k -coloring the complete graph on $k+1$ vertices, i.e., $\text{Hom}(K_{k+1}, K_k)$, then in [HMPS19] it is shown that $\mathcal{A}(\text{Hom}(K_{k+1}, K_k)) = 0$ for $k \leq 3$ while $\mathcal{A}(\text{Hom}(K_{k+1}, K_k)) \neq 0$ for all $k \geq 4$. In fact in [HMPS19] it is shown that for $k \geq 4$ these algebras contain k self-adjoint idempotents that sum to $(k+1)$ times the identity and that -1 can be written as a sum of squares of self-adjoint elements. These results show that these nonzero algebras possess no states and can have no representations as algebras of operators on a Hilbert space.

The proof that $\mathcal{A}(\text{Hom}(K_5, K_4)) \neq 0$, which is the algebra that we discussed in the section where we introduced game algebras, necessitates showing that $I \notin \mathcal{J}(\text{Hom}(K_5, K_4))$ and the only known proof of this fact is machine-assisted using programs for finding Gröbner bases in noncommutative algebras [HMPS19].

7.6. Locally Commuting Algebras for Graphs. Because the algebras of graph homomorphism games can be so intractable, there is also another algebra that has been studied. These algebras have the same generators and relations as for $\mathcal{A}(\text{Hom}(G_1, G_2))$ together with the additional relations that

$$(v, w) \in E_1 \implies e_{v,a}e_{w,b} = e_{w,b}e_{v,a}, \forall a, b \in V_2.$$

Thus, “locally” all of the elements of the generating PVM’s commute. See for example [HMPS19] where this local algebra is studied for graph coloring games.

7.7. Multiparty Graph Coloring. The graph coloring game has a natural m -party analogue. Because of the fact pointed out earlier that synchronous qc-correlations in more than two parties are local, one has that $\chi_{qc}(G) = \chi(G)$ as soon as there are 3 or more players. However, $\chi_{ns}(G)$ need not be equal to $\chi(G)$, but is known to increase to $\chi(G)$ as the number of players increases. There has been some interest in finding the least number of players, $m(G)$, needed to reach $\chi(G)$ and in characterizing the set of graphs G such that $\chi_{ns}(G) \leq 2$, in the m -player case.

8. THE GRAPH ISOMORPHISM GAME

There are many beautiful results and ongoing research about this game. It also has an important connection with the study of quantum groups. It was first introduced and studied in [AMR⁺19].

Given a graph $G = (V, E)$ we define a function

$$rel : V \times V \rightarrow \{-1, 0, +1\},$$

via

$$rel(x, y) = \begin{cases} -1, & (x, y) \in E, \\ 0, & x = y, \\ +1, & (x, y) \notin E \text{ and } x \neq y \end{cases}.$$

The matrix $S_G := (rel(x, y))_{x, y \in V}$ is called the *Siedel adjacency matrix* of G and plays an important role in graph theory.

Two graphs $G_i = (V_i, E_i)$, $i = 1, 2$ are *isomorphic* if there is a one-to-one, onto function $f : V_1 \rightarrow V_2$ such that

$$rel(x, y) = rel(f(x), f(y)), \forall x, y \in V_1.$$

. In this case we write $G_1 \simeq G_2$. If $card(V_1) = card(V_2) = n$ and we choose some enumeration of these two sets, then it is easily seen that the $n \times n$ matrix P that is one in the $(i, f(i))$ -entry and 0 elsewhere is a permutation matrix that intertwines the two Siedel adjacency matrices. Conversely, any permutation matrix that intertwines the two Siedel adjacency matrices defines a function that is a graph isomorphism.

The **graph isomorphism game** $Iso(G_1, G_2)$ is the game with input set $I = V_1$, output set $O = V_2$, and we will define it in terms of its winning set

$$W = \{(a, b, x, y) \in V_2 \times V_2 \times V_1 \times V_1 \mid rel(x, y) = rel(a, b)\}.$$

This was first introduced and studied in [AMR⁺19].

It is easy to see that this is a synchronous game and that when $card(V_1) = card(V_2)$, then $f : V_1 \rightarrow V_2$ is a perfect deterministic strategy if and only if f is a graph isomorphism. More generally, f is a perfect deterministic strategy if and only if $card(V_1) \leq card(V_2)$ and G_1 is isomorphic to the induced subgraph of G_2 on the subset $f(V_1)$.

When $card(V_1) = card(V_2)$, for $t = loc, q, qs, qa, qc, ns$ we write $G_1 \simeq_t G_2$ if and only if there exists a perfect density for this game in C_t .

For the rest of this section, we assume that $card(V_1) = card(V_2)$ for simplicity of exposition.

Lovasz proved that graphs G_1 and G_2 are isomorphic if and only if the number of graph homomorphisms from H to G_1 is equal to the number of graph homomorphisms from H to G_2 for all graphs H .

A beautiful result of [MR20] shows that $G_1 \simeq_{qc} G_2$ if and only if the number of graph homomorphisms from H to G_1 is equal to the number of graph homomorphisms from H to G_2 for all planar graphs H .

In [BCE⁺20] it is proven that $G_1 \simeq_{qc} G_2$ if and only if $\mathcal{A}(Iso(G_1, G_2)) \neq 0$.

There is also a beautiful connection between linear system games over the binary field and graph isomorphisms.

Given a binary system of equations $A\vec{x} = \vec{b}$, Arkhipov (see also [AMR⁺19]) constructed a graph $G_{A, \vec{b}}$ with the property that $G_{A, \vec{b}} \simeq G_{A, \vec{0}}$ if and only if the system of equations has a solution. So the graph for the non-homogeneous

is isomorphic to the graph for the homogeneous system if and only if the non-homogeneous has a solution.

In [AMR⁺19] it is proven that for $t = loc, q, qc$,

$$G_{A,b} \simeq_t G_{A,\vec{b}} \iff LCS(A, \vec{b}) \text{ has a perfect density in } C_t.$$

Later, after introducing the synchronous version of the $LCS(A, \vec{b})$ game it was shown that

$$G_{A,\vec{b}} \simeq_t G_{A,\vec{0}} \iff syncLCS(A, \vec{b}) \text{ has a perfect density in } C_t,$$

for $t = loc, q, qs, qa, qc$ [KPS18].

The fundamental orthogonality relations for the graph isomorphism game follow from the results in the paper [AMR⁺19], where it is shown that these relations are equivalent to the following conditions:

- (1) For each $x \in V_1$, $\{E_{x,a} : a \in V_2\}$ is an orthogonal family of projections summing to the identity.
- (2) For each $a \in V_2$, $\{E_{x,a} : x \in V_1\}$ is an orthogonal family of projections summing to the identity.
- (3) For each $x \in V_1$ and $a \in V_2$,

$$\sum_{\{x_1 : (x, x_1) \in E_1\}} E_{x_1, a} = \sum_{\{a_2 : (a, a_2) \in E_2\}} E_{x, a_2}.$$

This last relation is best visualized as follows. If we let A_G denote the usual adjacency matrix of a graph, then the third relation is that

$$A_{G_1}(E_{x,a}) = (E_{x,a})A_{G_2},$$

which is equivalent to

$$S_{G_1}(E_{x,a}) = (E_{x,a})S_{G_2}.$$

For another derivation of these relations in a language closer to these notes, see [BCE⁺20].

8.1. The Quantum Permutation Group. Any matrix of projections $E = (E_{x,a})$ satisfying conditions (1) and (2) from above is called variously a *magic unitary* or a *quantum permutation*. This wording is motivated by the fact that $E^*E = EE^* = I$ and if the $E_{x,a}$'s all belonged to an abelian C*-algebra, $E_{x,a} \in C(T)$ then at each point in the spectrum, the matrix $(E_{x,a}(t))$ would be a permutation.

The *quantum permutation group* $C^*(S_n^+)$ is the universal C*-algebra generated by n^2 projections, $\{e_{i,j} : 1 \leq i, j \leq n\}$, satisfying (1) and (2) above with $V_1 = V_2 = \{1, \dots, n\}$.

Given a graph G_1 on n vertices, the completion of $\mathcal{A}(Iso(G_1, G_1))$ is an often studied quotient of the quantum permutation group for which there are currently many unanswered questions. It is a particularly nice quotient that satisfies the axioms to be a *subgroup of the quantum permutation group* (recall that quantum groups are a type of dual object to groups). See [BKS23, dICDN20] for some of these related problems.

In [BCE⁺20] modules over $\mathcal{A}(\text{Iso}(G_i, G_i))$ are studied and it is shown that $G_1 \sim_{qc} G_2$ if and only if there is a certain Morita-type equivalence between the modules over $\mathcal{A}(\text{Iso}(G_1, G_1))$ with the modules over $\mathcal{A}(\text{Iso}(G_2, G_2))$.

8.2. Quantum Analogues of Doubly Stochastic Matrices. Recall that an $n \times n$ matrix $P = (p_{i,j})$ is *doubly stochastic* if $p_{i,j} \geq 0$ and the rows and columns sum to 1. A famous result of Birkhoff is that the doubly stochastic matrices are the convex hull of the set of permutation matrices.

A *quantum doubly stochastic operator* is an $n \times n$ matrix of operators $P = (P_{i,j})$ where $P_{i,j} \in B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ satisfying $P_{i,j} \geq 0$ and that the rows and columns sum to $I_{\mathcal{H}}$. Even for $\mathcal{H}$ finite dimensional the quantum analogue of Birkhoff's result fails: the quantum doubly stochastic operators are not the closed convex hull of the quantum permutations. See [dlCDN20].

One might expect a dilation version of Birkhoff's theorem, i.e., that given a doubly stochastic operator $P = (P_{i,j})$, $P_{i,j} \in B(\mathcal{H})$, that there exists a unital completely positive map $\phi : C^*(S_n^+) \rightarrow B(\mathcal{H})$ with $\phi(e_{i,j}) = P_{i,j}$. The Stinespring dilation of this map would then dilate the doubly stochastic operator to a quantum permutation. Even for $\mathcal{H} = \mathbb{C}^2$ this fails. For example, consider:

$$\begin{aligned} P_{1,1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{1,2} = \begin{pmatrix} 0 & 0 \\ 0 & 3/4 \end{pmatrix}, \quad P_{1,3} = \begin{pmatrix} 0 & 0 \\ 0 & 1/4 \end{pmatrix}, \\ P_{2,1} &= \begin{pmatrix} 0 & 0 \\ 0 & 3/4 \end{pmatrix}, \quad P_{2,2} = \begin{pmatrix} 1/2 & a \\ a & 1/8 \end{pmatrix}, \quad P_{2,3} = \begin{pmatrix} 1/2 & -a \\ -a & 1/8 \end{pmatrix}, \\ P_{3,1} &= \begin{pmatrix} 0 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad P_{3,2} = \begin{pmatrix} 1/2 & -a \\ -a & 1/8 \end{pmatrix}, \quad P_{3,3} = \begin{pmatrix} 1/2 & a \\ a & 5/8 \end{pmatrix}. \end{aligned}$$

It is easily checked that these matrices define a 3×3 quantum doubly stochastic matrix for any $-1/4 \leq a \leq +1/4$. As long as $a \neq 0$, one can prove that there is no unital completely positive map $\phi : C^*(S_3^+) \rightarrow M_2$ with $\phi(e_{i,j}) = P_{i,j}, \forall i, j$.

There are still many open problems concerning characterizing the extreme points of the set of quantum doubly stochastic operators and giving necessary and sufficient conditions for a quantum doubly stochastic matrix to induce a unital completely positive map on the quantum symmetric group. For more results in this direction, see [dlCDN20].

8.3. Bisynchronous Games. A synchronous game $\mathcal{G} = (I, O, V)$ is called *bisynchronous* provided that in addition to being synchronous it satisfies

$$V(a, a, x, y) = 0, \quad \forall x \neq y.$$

Thus, in addition to the synchronous requirement that whenever the players are given the same question they must give the same response, is the requirement that whenever they give the same answer, then they must have been

asked the same question. This family of games was introduced in [PR21], motivated by the fact that the graph isomorphism game is bisynchronous.

A *bisynchronous density* is one satisfying

$$p(a, b|x, x) = 0, \forall a \neq b \text{ and } p(a, a|x, y) = 0, \forall x \neq y.$$

The set of bisynchronous densities of the various types are denoted $C_t^{bs}(n, k)$. It is shown in [PR21] that a density in $C_{qc}^{bs}(n, n)$ is bisynchronous if and only if there is a trace τ on the quantum permutation group $C^*(S_n^+)$ such that

$$p(a, b|x, y) = \tau(e_{x,a}e_{y,b}),$$

with corresponding results for $C_t^{bs}(n, n)$.

The structure of these sets of densities is not very well understood. In particular, it is not known if $C_{qa}^{bs}(n, n)$ is the closure of $C_q^{bs}(n, n)$.

Bisynchronous densities are also related to the theory of *factorizable maps* in the sense of C. Anantharaman-Delaroche [Ana06] and the theory of their *ancillas* as introduced and studied by U. Haagerup, M. Musat, and M. Rørdam [HM11, HM15, MR20]. See also [HLP⁺18].

Given a bisynchronous density $p(a, b|x, y) \in C_t^{bs}(n, n)$ [PR21] proves that if we define a linear map $\Phi : M_n \rightarrow M_n$ on the spanning matrix units by

$$\Phi(E_{x,y}) = \sum_{a,b} p(a, b|x, y) E_{a,b},$$

then such maps are always factorizable with the type of ancilla needed depending on t . In particular, the problem about the closure of $C_q^{bs}(n, n)$ is related to whether or not factorizable maps with ancilla $\mathcal{R}^\omega$ are limits, in an appropriate sense, of factorizable maps with finite dimensional ancilla.

9. OPERATIONS ON GAMES

In group theory, we have many ways of producing new groups from given groups. There are many parallels in the theory of games. One of the keys to the proof of the negation of Connes' Embedding Problem in [JNV⁺20] was certain ways to obtain new games from old games that had known consequences for the values of the games.

In this section we discuss a few of these to serve as an introduction to the literature.

The first operation is the *product of games*. Given two games $\mathcal{G}_i = (X_i, Y_i, A_i, B_i V_i), i = 1, 2$ their product $\mathcal{G}_1 \times \mathcal{G}_2$ is the game with input sets $X := X_1 \times X_2, Y = Y_1 \times Y_2$, and output sets $A := A_1 \times A_2, B = B_1 \times B_2$ and rule function,

$$V : X \times X \times O \times O \rightarrow \{0, 1\} = \mathbb{Z}_2,$$

given by

$$V((x_1, x_2), (y_1, y_2), (a_1, a_2), (b_1, b_2)) = V_1(x_1, y_1, a_1, b_1) V_2(x_2, y_2, a_2, b_2),$$

where the product is in $\mathbb{Z}_2$. Thus, they win if and only if $V_1(x_1, y_1, a_1, b_1) = 1$ and $V_2(x_2, y_2, a_2, b_2) = 1$, that is if and only if they win both games. It is customary to write $V = V_1 \times V_2$.

It is not hard to see that for $t = loc, q, qa, qs, qa, qc, ns$ the if $p(a_i, b_i | x_i, y_i) \in C_t(|X_i|, |Y_i|, |A_i|, |B_i|)$ then

$$p((a_1, a_2), (b_1, b_2) | (x_1, x_2), (y_1, y_2)) := p_1(a_1, b_1 | x_1, y_1) p_2(a_2, b_2 | x_2, y_2)$$

is in $C_t(|X_1 \times X_2|, |Y_1 \times Y_2|, |A_1 \times A_2|, |B_1 \times B_2|)$. Consequently, if each game has a perfect t-strategy, then their product does also.

This also shows that if we assume that each game has a prior distribution π_1, π_2 on inputs and we define a distribution on inputs $\pi := \pi_1 \times \pi_2$ by $\pi((x_1, x_2), (y_1, y_2)) = \pi_1(x_1, y_1) \pi_2(x_2, y_2)$, then

$$\omega_t(\mathcal{G}_1 \times \mathcal{G}_2, \pi_1 \times \pi_2) \geq \omega_t(\mathcal{G}_1, \pi_1) \cdot \omega_t(\mathcal{G}_2, \pi_2).$$

This is summarized by saying that the value is *supermultiplicative*. Feige and Kilian [FK00] were the first to show that this inequality can be strict.

The *parallel repetition* of a game $(\mathcal{G}, \pi)$ is the game $(\mathcal{G}^n, \pi^n)$ obtained by taking the product of the game and density with itself n times and there is a great deal of research determining how the sequence $\omega_t(\mathcal{G}^n, \pi^n)$ behaves. Since the value is supermultiplicative, there is no a priori guarantee that when $\omega_t(\mathcal{G}, \pi) < 1$ that this sequence of numbers even tends to 0.

Raz [Raz98] proved that $\omega_{loc}(\mathcal{G}^n, \pi^n)$ decays exponentially and Holenstein [Hol09] proved that $\omega_{ns}(\mathcal{G}^n, \pi^n)$ decays exponentially. Yuen [Yue16] proved that $\omega_q(\mathcal{G}^n, \pi^n)$ decays inverse polynomially, and it is still unknown if this decay could also be exponential.

It is currently unknown if $\omega_{qc}(\mathcal{G}^n, \pi^n)$ even tends to 0.

For synchronous values of games even less is known. The crucial cases are when $\omega_t^s(\mathcal{G}, \pi) < \omega_t(\mathcal{G}, \pi) = 1$. In [HMN⁺24] an example is given of a nonsynchronous game with equal input and output sets such that

$$\omega_{loc}^s(\mathcal{G}^n, \pi^n) = \omega_{qc}^s(\mathcal{G}^n, \pi^n) = 1 - \frac{1}{2^n},$$

so the value is monotone increasing!

In [MPTW23] the first example is given of a synchronous game for which the synchronous value is strictly supermultiplicative. This paper also proves that for synchronous games $\mathcal{A}(\mathcal{G}_1 \times \mathcal{G}_2) = \mathcal{A}(\mathcal{G}_1) \otimes \mathcal{A}(\mathcal{G}_2)$, which is a somewhat surprising since if say $\mathcal{G}_i$ has n_i inputs and k_i outputs, then $\mathcal{A}(\mathcal{G}_1 \times \mathcal{G}_2)$ is a quotient of the algebra of the free group $\mathbb{F}(n_1 n_2, k_1 k_2) = \mathbb{F}(n_1, k_1 k_2) \star \mathbb{F}(n_2, k_1 k_1)$.

Two new operations on games, *anchoring* and *fortification*, were introduced and studied in [BVY15, BVY16, BVY17] as a means to guarantee exponential decay. In particular, given $(\mathcal{G}, \pi)$ its anchor and fortification are new games $(\mathcal{H}, \rho)$ with the properties that:

- $\omega_q(\mathcal{G}, \pi) = 1 \implies \omega_q(\mathcal{H}, \rho) = 1$,
- $\omega_q(\mathcal{G}, \pi) < 1 \implies \omega_q(\mathcal{H}^n, \rho^n) \leq \exp(-\Omega(n))$.

In [JNV⁺20] several other operations that obtain new games for given games are introduced, including an operation that starts with a sequence of games and a Turing machine to produce a new game that allows them to fine tune the number of inputs, number of outputs, and the value.

For special families of games there are also often special operations for combining these games to obtain a game in the same family. One such example is XOR games.

First note that the product of two XOR games is not an XOR game. In fact the product is not even a game with binary answers. An operation on XOR games, studied in [CSUU06], produces an XOR game. The *XOR of XOR games* $\mathcal{G}_1$ and $\mathcal{G}_2$ with densities π_1, π_2 and rule functions f_1 and f_2 , denoted by $\mathcal{G}_1 \oplus \mathcal{G}_2$, is the XOR game $(I_1 \times I_2, \{0, 1\}, V)$ with distribution $\pi_1 \times \pi_2$ and rule function defined by $V((x_1, x_2), (y_1, y_2), a, b) = 1$ iff $a + b = f_1(x_1, y_1) + f_2(x_2, y_2)$ in $\mathbb{Z}_2$. The XOR of more than two games is defined similarly.

The *bias*, of a game defined as,

$$\epsilon_t(\mathcal{G}, \pi) := 2\omega_t(\mathcal{G}, \pi) - 1.$$

It is proven in [CSUU06] that the bias of the XOR of XOR games behaves multiplicatively. However, in [HMN⁺24] it is shown that when the games are synchronous the synchronous bias

$$\epsilon_t^s(\mathcal{G}, \pi) := 2\omega_t^s(\mathcal{G}, \pi) - 1,$$

need not be multiplicative.

10. GAMES WITH QUANTUM QUESTIONS AND ANSWERS

So far, we have considered non-local games where the players are asked questions from finite question sets, and give answers from finite answer sets. Several generalizations of this setting have been considered in the literature, such as games with three or more players, or games where the question or answer sets are allowed to be infinite [MV18]. In this section, we consider games where either the question sets, answer sets, or both, are quantized. One way to think of quantum questions and answers is that, instead of the players receiving a classical question pair $(x, y) \in X \times Y$, they receive a *quantum state* from the referee in the registers X and Y , and they send a quantum state back to the referee in the registers A and B .

Operationally, one can consider such games as follows. The referee prepares a quantum state, i.e., a density matrix ρ_{XY} acting on $\mathbb{C}^X \otimes \mathbb{C}^Y$ and sends this state to Alice and Bob. Alice has access to the register X and Bob has access to the register Y . If we let M_X denote the $X \times X$ matrices, then $\rho_{XY} \in M_X \otimes M_Y$. The players will return a density matrix acting on $\mathbb{C}^A \otimes \mathbb{C}^B$, i.e., in $M_A \otimes M_B$. The fact that this is a quantum game is reflected in the requirement that the players apply a quantum channel, i.e., a completely positive trace-preserving map, $\Gamma : M_X \otimes M_Y \rightarrow M_A \otimes M_B$, to obtain their output. Thus, a strategy becomes a particular quantum

channel. The referee receives the state $\rho_{AB} = \Gamma(\rho_{XY})$ on the registers A and B . The *rules* of the game are encoded in a pair of projections $\{\Pi_{\text{acc}}, I - \Pi_{\text{acc}}\} \in M_A \otimes M_B$. The referee performs the *measurement* on ρ_{AB} given by $\{\text{Tr}(\Pi_{\text{acc}}\rho_{AB}), \text{Tr}((I - \Pi_{\text{acc}})\rho_{AB})\}$ and this pair of numbers represent the respective probabilities that they win or lose.

The quantum channel Γ that the players apply is assumed to satisfy “no-signalling conditions”, as with classical non-local games. Formally, this amounts to requiring that, for all $\rho_X \in M_X$ and $\rho_Y \in M_Y$, one has $\text{Tr}_A(\Gamma(\rho_X \otimes \rho_Y)) = 0$ whenever $\text{Tr}(\rho_X) = 0$ and $\text{Tr}_B(\Gamma(\rho_X \otimes \rho_Y)) = 0$ whenever $\text{Tr}(\rho_Y) = 0$. Here, $\text{Tr}_A : M_A \otimes M_B \rightarrow M_B$ and $\text{Tr}_B : M_A \otimes M_B \rightarrow M_A$ represent the *partial traces* given on elementary tensors by $\text{Tr}_A(C \otimes D) = \text{Tr}(C)D$ and $\text{Tr}_B(C \otimes D) = \text{Tr}(D)C$, respectively.

In this setting, one can define local, quantum, quantum approximate and quantum commuting correlations, as in the classical question and answer setting. The key difference is the operators the players possess. Alice’s POVMs $\{E_{x,a}\}_{x \in X, a \in A}$ in $\mathcal{B}(\mathcal{H})$ are replaced by a single *stochastic operator matrix* $P \in M_X \otimes M_A \otimes \mathcal{B}(\mathcal{H})$; that is, $P \geq 0$ and $\text{Tr}_A(P) = I_X \otimes I_{\mathcal{H}}$. Similarly, Bob possesses a stochastic operator matrix $Q \in \mathcal{B}(\mathcal{H}) \otimes M_Y \otimes M_B$. The commuting operator framework assumption requires that P and Q commute when considered as operators on the quintuple tensor product $M_X \otimes M_A \otimes \mathcal{B}(\mathcal{H}) \otimes M_Y \otimes M_B$. This condition is equivalent to the block entries $P_{x,x',a,a'}, x, x' \in X, a, a' \in A$, of P , commuting with each of the block entries $Q_{y,y',b,b'}, y, y' \in Y, b, b' \in B$, of Q . If the players share a prepared state $\psi \in \mathcal{H}$, then the map $\Gamma : M_X \otimes M_Y \rightarrow M_A \otimes M_B$ given by

$$\Gamma(E_{x,x'} \otimes E_{y,y'}) = \sum_{\substack{a,a' \in A \\ b,b' \in B}} \langle P_{x,x',a,a'} Q_{y,y',b,b'} \psi, \psi \rangle E_{a,a'} \otimes E_{b,b'}$$

is a quantum channel—we can regard this Γ as a *quantum commuting correlation* with quantum inputs and quantum outputs. Similarly, a *quantum correlation* in this setting is one where the Hilbert space $\mathcal{H}$ decomposes as a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and Alice’s stochastic operator matrix P acts only on $\mathbb{C}^X \otimes \mathbb{C}^A \otimes \mathcal{B}(\mathcal{H}_A)$, while Bob’s acts on $\mathcal{B}(\mathcal{H}_B) \otimes \mathbb{C}^Y \otimes \mathbb{C}^B$; then the products $P_{x,x',a,a'} Q_{y,y',b,b'}$ are replaced with tensor products. As in the classical case, *quantum approximate correlations* are those quantum channels that are pointwise limits of quantum channels arising from quantum correlations. Lastly, *local correlations* are those that are convex combinations of tensor products of quantum channels—that is, such Γ are of the form $\Gamma = \sum_{i=1}^k \lambda_i \Phi_i \otimes \Psi_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, and for each i , the maps $\Phi_i : M_X \rightarrow M_A$ and $\Psi_i : M_Y \rightarrow M_B$ are quantum channels. Equivalently, if written using stochastic operator matrices, local correlations are those where all the block entries of both stochastic operator matrices commute with each other [BHTT23a].

Correlations with classical questions and answers naturally fit into this more general framework. Indeed, suppose that $p \in C_{qc}(X, Y, A, B)$ is given

by $p(a, b|x, y) = \langle E_{x,a} F_{y,b} \psi, \psi \rangle$, where $\psi \in \mathcal{H}$ is a unit vector and $\{E_{x,a} : 1 \leq x \leq |X|, 1 \leq a \leq |A|\}$ is Alice's $(|X|, |A|)$ -POVM and $\{F_{y,b} : 1 \leq y \leq |Y|, 1 \leq b \leq |B|\}$ is Bob's $(|Y|, |B|)$ -POVM on $\mathcal{H}$ such that $[E_{x,a}, F_{y,b}] = 0$ for all x, y, a, b . Then, letting $\mathcal{D}_X$ be the diagonal subalgebra of M_X with respect to the canonical basis $\{e_x : x \in X\}$, the map $\Gamma : \mathcal{D}_X \otimes \mathcal{D}_Y \rightarrow \mathcal{D}_A \otimes \mathcal{D}_B$ given by

$$\Gamma(e_x \otimes e_y) = \sum_{a \in A, b \in B} p(a, b|x, y) e_a \otimes e_b$$

is a (classical) channel and reflects the probability that the players respond with answers a and b , given that they received x and y , respectively.

The perspective of quantum channels is not the only perspective for correlations in the quantum input and/or output setting. Work of A. Bochniak, P. Kasprzak and P. Soltan [BKS23] considers correlations arising from unital, completely positive maps. The two are related in that unital completely positive maps are dual to quantum channels.

Sometimes, in this setting, non-local games discussed earlier on are referred to as “classical” non-local games, or non-local games with “classical questions and answers”. Such games fall naturally into this framework by requiring that the density ρ_{XY} have a singular value decomposition in terms of the canonical basis vectors for $\mathbb{C}^X \otimes \mathbb{C}^Y$ —that is, that $\rho_{XY} = \sum_{(i,j) \in X \times Y} p_{ij} e_i e_i^* \otimes e_j e_j^*$, where $p_{ij} \geq 0$ and $\sum_{(i,j) \in X \times Y} p_{ij} = 1$. Then the density ρ_{XY} represents the players receiving the classical question pair $(i, j) \in X \times Y$ with probability p_{ij} , in a non-local game with classical questions. Classical answers, then, can be modelled by the referee receiving a density in a similar manner. Formally, we think of a non-local game with classical questions and classical answers as a quantum to quantum game, where we replace M_X, M_Y, M_A and M_B with the diagonal subalgebras $\mathcal{D}_X, \mathcal{D}_Y, \mathcal{D}_A$ and $\mathcal{D}_B$ of the corresponding matrix algebras.

There are different variants of allowing quantum questions and/or answers for these games. We can describe these variants based on what the density ρ_{XY} is, and what the referee's binary measurement system is. We call a game

- *Classical to classical* if $\rho_{XY} \in \mathcal{D}_X \otimes \mathcal{D}_Y$ and $\Pi_{\text{acc}}((\mathcal{D}_X \otimes \mathcal{D}_Y)^\perp) = \{0\}$;
- *Quantum-to-classical* if $\rho_{XY} \in M_X \otimes M_Y$ and $\Pi_{\text{acc}}((\mathcal{D}_X \otimes \mathcal{D}_Y)^\perp) = \{0\}$;
- *Classical-to-quantum* if $\rho_{XY} \in \mathcal{D}_X \otimes \mathcal{D}_Y$, and
- *Quantum-to-quantum* otherwise.

10.1. Quantum XOR games. As XOR games are one of the fundamental classes of non-local games with classical questions and answers, the class of quantum XOR games are one of the fundamental classes of quantum-to-classical games and already showcase many of the differences that occur when allowing quantum questions or answers. These games were introduced by O. Regev and T. Vidick [RV15]. The similarity between XOR games and

quantum XOR games is that the answer sets for the players are $\{0, 1\}$, and the players win or lose each round of the game based only on the parity of their bits, depending on the question state.

For quantum XOR games, it can be easier to view the question state of the game in terms of an ensemble of states, rather than a single density matrix. Given the input density ρ_{XY} , one can find unit vectors $\{\varphi_{i,j}\}_{(i,j) \in X \times Y}$ and non-negative scalars $p_{ij} \geq 0$ such that $\sum_{i,j} p_{ij} = 1$ and

$$\rho_{XY} = \sum_{(i,j) \in X \times Y} p_{ij} \varphi_{ij} \varphi_{ij}^*,$$

where the collection $\{\varphi_{ij} : (i,j) \in X \times Y\}$ is perfectly distinguishable; i.e., φ_{i_1,j_1} and φ_{i_2,j_2} are orthogonal whenever $(i_1, j_1) \neq (i_2, j_2)$. Then the rules for the XOR of the players' answers depends on the sign associated with the unit vector $\varphi_{ij} \in \mathbb{C}^X \otimes \mathbb{C}^Y$. If the associated sign is $c_i \in \{-1, 1\}$ and the players received φ_{ij} , then their answers a and b must satisfy $(-1)^{a+b} = c_i$. If all of the vectors φ_{ij} are product states, then by applying local unitaries on the register spaces, one obtains a classical XOR game.

For classical XOR games, the optimal value for the game using quantum strategies is attained in the model C_q , and the ratio of the q -value of the game to the classical value is bounded by a constant [Tsi93]. For quantum XOR games, the situation is very different. For example, for $n \geq 2$, the " T_n -game" in [RV15] does not have this property. For this game, players receive one of the states $\psi_0 = \frac{1}{\sqrt{2}} e_0 \otimes e_0 + \frac{1}{\sqrt{2n}} \sum_{i=1}^n e_i \otimes e_i$ or $\psi_1 = \frac{1}{\sqrt{2}} e_0 \otimes e_0 - \frac{1}{\sqrt{2n}} \sum_{i=1}^n e_i \otimes e_i$, each with probability $\frac{1}{2}$. If the players respond with bits $a, b \in \{0, 1\}$ respectively, then they must satisfy $a = b$ if they received ψ_0 , and $a \neq b$ if they received ψ_1 .

The key property of a winning strategy for the T_n game is *perfect embezzlement of entanglement*, which was examined in work of R. Cleve, L. Liu and the second author [CLP17] for T_2 , although it naturally generalizes to T_n . The work of [CLP17] shows that winning the game in the commuting operator framework with probability 1 is equivalent to the players possessing unitaries $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ with $(U \otimes I_n)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_n)$, and a shared state $\psi \in \mathcal{H}$ satisfying

$$(U \otimes I_n)(I_n \otimes V)(e_1 \otimes \psi \otimes e_1) = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes \psi \otimes e_i.$$

Such a protocol is called *perfect embezzlement* (of entanglement) because the entanglement of the state on the players' registers is greater than what they started with, without disturbing the shared vector ψ . Perfect embezzlement of entanglement, in this sense, can be done approximately in finite dimensions [vdH03], but cannot be done perfectly in finite dimensions or even in a tensor product framework where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and U acts on $\mathbb{C}^n \otimes \mathcal{H}_A$ and V on $\mathcal{H}_B \otimes \mathbb{C}^n$ [CLP17]. In terms of the T_n quantum XOR game, this means that the quantum value of the game is 1, even though

there is no q -strategy that yields winning probability exactly 1—that is, the qa value of the game is 1. On the other hand, the classical value of T_n is $\frac{1}{2}(1 + \frac{1}{\sqrt{n}})$ [RV15]. If one uses the notion of winning *bias*, as in classical XOR games, then the quantum bias of T_n is 1, while the classical bias of T_n is $\frac{1}{\sqrt{n}}$ [RV15]. Thus, the ratio of the quantum bias to the classical bias for T_n tends to infinity as n tends to infinity. In contrast, the ratio of the quantum bias to the classical bias for XOR games is bounded above by the real Grothendieck constant [Tsi80, RV15]. This, as noted in [RV15], follows directly from the Grothendieck inequality [Gro53].

More is true about quantum XOR games. Whereas optimal values of classical XOR games do not distinguish between the q , qa and qc models, quantum XOR games already distinguish between the q and qa models, using the example of T_n . Due to work of the first author [Har17] and the negative resolution of Connes' embedding problem [JNV⁺20], there exists a quantum XOR game with a higher qc value than qa value. Another key difference is that classical XOR games satisfy parallel repetition [CSUU06], while there exist quantum XOR games that violate parallel repetition for both the local and quantum values [RV15]. Thus, at first glance, non-local games with quantum questions or answers appear to be much more general than non-local games with classical questions and answers.

10.2. Quantum games for quantum graphs. Another recent avenue in quantum non-local games concerns quantum graphs. As there are several approaches to quantum graphs, one also obtains several different approaches to quantum non-local games that are related to quantum graphs.

One approach is by considering quantum graphs as a quantum form of a relation [Wea21]. In this setting, a quantum graph is a triple $(\mathcal{S}, \mathcal{M}, M_n)$, where $\mathcal{S}$ is an operator subsystem of M_n and $\mathcal{M} \subseteq M_n$ is a finite-dimensional C^* -algebra such that $\mathcal{S}$ is a bimodule over the relative commutant $\mathcal{M}'$. These naturally generalize classical graphs—a classical graph G has an associated graph operator system $\mathcal{S}_G = \text{span}\{E_{ij} : i = j \text{ or } i \sim_G j\} \subseteq M_{V(G)}$, which is a bimodule over the diagonal subalgebra $\mathcal{D}_{V(G)} = \mathcal{D}'_{V(G)}$.

In this setting of quantum graphs, work of M. Brannan, P. Ganesan and the first author considered the graph homomorphism game from a quantum graph $(\mathcal{S}, \mathcal{M}, M_n)$ to a classical graph G [BGH22]. This game is another example of a quantum-to-classical non-local game with $X = Y = \{1, \dots, n\}$ and $A = B = V(G)$. With classical outputs, the players' stochastic operator matrices $P \in M_n \otimes \mathcal{D}_{V(G)} \otimes \mathcal{B}(\mathcal{H})$ and $Q \in M_n \otimes \mathcal{D}_{V(G)} \otimes \mathcal{B}(\mathcal{H})$ are precisely given by a single POVM in $M_n \otimes \mathcal{B}(\mathcal{H})$ for each player, with $|V(G)|$ outputs [BGH22]. This game, as in the classical graph homomorphism game, satisfies a form of synchronicity—for winning strategies, Bob's single POVM can be obtained uniquely from Alice's POVM, and the state vector that the players share induces a tracial state on the C^* -algebras generated by each of the players' block operators. Winning strategies for the homomorphism game from $(\mathcal{S}, \mathcal{M}, M_n)$ to G are given precisely by a PVM $\{P_a\}_{a=1}^c$ in $\mathcal{M} \otimes \mathcal{N}$, for

some finite von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, such that $P_a((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes 1_{\mathcal{N}})P_b = 0$ whenever $a, b \in V(G)$ and $a \not\sim_G b$ [BGH22]. Since such a PVM arises from a unital completely positive map from $\mathcal{D}_{V(G)}$ to $\mathcal{M} \otimes \mathcal{N}$, one can alternatively describe the winning strategy via Choi-Kraus operators; in this setting, one has operators $F_i \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}, \mathbb{C}^{V(G)})$ such that $\sum_i F_i^* F_i = I$, and the two conditions

$$F_i((\mathcal{S} \cap (\mathcal{M}')^\perp) \otimes 1_{\mathcal{N}})F_j^* \subseteq \mathcal{S}_G \cap (D_{V(G)})^\perp, \forall i, j$$

and

$$F_i(\mathcal{M}' \otimes 1_{\mathcal{N}})F_j^* \subseteq \mathcal{D}_{V(G)}, \forall i, j$$

are satisfied. We refer the reader to [CKLT16] for more information in the case of infinitely many Choi-Kraus operators. These conditions are akin to the notion of a *pushforward* of a quantum graph by a quantum channel in the sense of N. Weaver [Wea21], where the pushforward also is a pushforward of $\mathcal{M}'$ to $\mathcal{D}_{V(G)}$.

In the case when $(\mathcal{S}, \mathcal{M}, M_n)$ is a classical graph, this criterion is equivalent to the existence of a winning *qc* strategy for the usual graph homomorphism game. In the *qa* setting, $\mathcal{N}$ can be replaced by $\mathcal{R}^{\mathcal{U}}$; in the *q* setting it can be replaced by M_d for some d , and in the local setting, one can arrange to have $\mathcal{N} = \mathbb{C}$.

When one allows quantum graphs, this game already possesses properties not seen in the classical setting. For example, one can define the *t-chromatic number*, for $t \in \{loc, q, qa, qc\}$, of the quantum graph $(\mathcal{S}, \mathcal{M}, M_n)$, as the smallest $c \in \mathbb{N}$ such that the homomorphism game $(\mathcal{S}, \mathcal{M}, M_n) \rightarrow K_c$ has a winning *t*-strategy. for the *quantum complete graph* $(M_n, \mathcal{M}, M_n)$ (a quantized version of the complete graph on n vertices, with the vertex set replaced by the “quantum set” $\mathcal{M}$), one has $\chi_q((M_n, \mathcal{M}, M_n)) = \dim(\mathcal{M})$, extending the classical graph result [BGH22]. However, if $\mathcal{M}$ is not abelian, then there is no $c \in \mathbb{N}$ for which the homomorphism game $(M_n, \mathcal{M}, M_n) \rightarrow K_c$ has a winning local strategy, so $\chi_{loc}((M_n, \mathcal{M}, M_n)) = \infty$. This reflects the idea that one should not be able to classically color a quantum graph, without the use of entanglement, and is closely related to earlier work of D. Stahlke [Sta16].

Alternatively, one can approach quantum graphs by considering classical graphs as traceless, self-adjoint operator subspaces of M_n , by associating to a classical graph G the operator space $\mathcal{T}_G = \text{span}\{E_{ij} : i \sim_G j\}$. The key difference in this approach is that it ignores the bimodule structure that graph operator systems have over the diagonal subalgebra, and instead focuses on the operator space structure. In this setting, winning strategies for the graph homomorphism and graph isomorphism games have similar descriptions via pushforwards [BHTT23a, BHTT23b]. The key difference is that the pushforward of commutants of the “quantum vertex sets” need not occur [BHTT23a, BHTT23b]. This omission is not surprising, since the traceless operator space approach to quantum graphs ignores the bimodule properties of the operator space. That being said, in the last few years there has been

a theory developed for homomorphism and isomorphism games for quantum graphs via traceless self-adjoint operator spaces. Winning strategies satisfy a condition known as *concurrency*, which forces the players' strategy to arise from a tracial state as with synchronous strategies. We refer the reader to [TT24, BHTT23a, BHTT23b] for more information.

We note that, while both of these viewpoints provide natural approaches to quantum graph coloring games, defining a quantum independence number for quantum graphs appears to be more challenging. Part of the issue stems from the difficulty in effectively defining the complement of a graph in the setting of quantum graphs. We refer the reader to [KM19] for more information, along with the recent work [CM23] for more information.

Reconciling these different kinds of homomorphism and isomorphism, depending on the definition of quantum graph used and the definition of the non-local game used, is still an active area of research. We refer the reader to [BHTT23b, Daw24] for more information.

REFERENCES

- [AHO97] F. Alizadeh, J.A. Haeberly, and M.L. Overton, *Complementarity and non-degeneracy in semidefinite programming*, Mathematical Programming **77** (1997), no. 2, 111–128.
- [Ana06] C. Anantharaman-Delaroche, *On ergodic theorems for free group actions on noncommutative spaces*, Probability Theory and Related Fields **135** (2006), 520–546.
- [ALM⁺98] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy, *Proof verification and the hardness of approximation problems*, Journal of the ACM **45** (1998), no. 3, 501–555.
- [AS98] S. Arora and S. Safra, *Probabilistic checking of proofs: A new characterization of NP*, Journal of the ACM **45** (1998), no. 1, 70–122.
- [AMR⁺19] A. Atserias, L. Mančinska, D.E. Roberson, R. Sámal, S. Severini, and A. Varvitsiotis, *Quantum and non-signalling graph isomorphisms*, Journal of Combinatorial Theory, Series B **136** (2019), 289–328.
- [AHKS06] D. Avis, J. Hagesawa, Y. Kikuchi, and Y. Sasaki, *A quantum protocol to win the graph coloring game on all hadamard graphs*, IEICE Trans. Fundam. electron. Commun. Comput. Sci., E89-A(5):1378–1381 (2006).
- [BFL91] L. Babai, L. Fortnow, and C. Lund, *Non-deterministic exponential time has two-prover interactive protocols*, Computational Complexity **1** (1991), 3–40.
- [BVY15] M. Bavarian, T. Vidick, and H. Yuen, *Anchored parallel repetition for nonlocal games*, preprint (arXiv:1509.07466v1) (2015).
- [BVY16] ———, *Parallel repetition via fortification: analytic view and the quantum case*, preprint (arXiv:1603.05349) (2016).
- [BVY17] ———, *Hardness amplification for entangled games via anchoring*, STOC'17: Proceedings of the 49th annual ACM SIGACT symposium on theory of computing, 2017, pp. 303–316.
- [Bel64] J.S. Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics **1** (1964), no. 3, 195–200.
- [BGKW88] M. Ben-Or, S. Goldwasser, J. Kilian, and A. Wigderson, *Multi-prover interactive proofs: How to remove intractability assumptions*, ACM Symposium on Theory of Computing, 1988, pp. 113–131.
- [BHK23] A. Bene Watts, J.W. Helton, and I. Klep, *Noncommutative nullstellensatz and perfect Games*, Annales Henri Poincaré **24** (2023), 2183–2239.

- [Boc91] F. Boca, *Free products of completely positive maps and spectral sets*, J. Funct. Anal. **97** (1991), 251–263.
- [BKS23] A. Bochniak, P. Kasprzak, and P.M. Soltan, *Quantum correlations on quantum spaces*, International Mathematics Research Notices **2023** (2023), no. 14, 12400–12440.
- [BCE⁺20] M. Brannan, A. Chirvasitu, K. Eifler, S.J. Harris, V.I. Paulsen, X. Su, and M. Wasilewski, *Bigalois extensions and the graph isomorphism game*, Communications in Mathematical Physics **375** (2020), 1777–1809.
- [BGH22] M. Brannan, P. Ganesan, and S.J. Harris, *The quantum-to-classical graph homomorphism game*, Journal of Mathematical Physics **63** (2022), 112204.
- [BHTT23a] M. Brannan, S.J. Harris, I.G. Todorov, and L. Turowska, *Synchronicity for quantum non-local games*, Journal of Functional Analysis **284** (2023), no. 2, 109738.
- [BHTT23b] ———, *Quantum no-signalling bicorrelations*, preprint (arXiv:2302.04268) (2023).
- [BO08] N. Brown and N. Ozawa, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.
- [CNM⁺07] P.J. Cameron, M.W. Newman, A. Montanaro, S. Severini, and A. Winter, *On the quantum chromatic number of a graph*, The electronic journal of combinatorics **14** (2007).
- [CL15] V. Capraro and M. Lupini, *Introduction to Sofic and Hyperlinear Groups and Connes’ Embedding Conjecture*, Lecture Notes in Mathematics, vol. 2136, Springer Cham, Switzerland, 2015.
- [CWY15] K.M. Chung, X. Wu, and H. Yuen, *Parallel repetition for entangled k-player games via fast quantum search*, CCC ’15: Proceedings of the 30th Conference on Computational Complexity, 2015, pp. 512–536.
- [CHSH69] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, *Proposed experiment to test local hidden-variable theories*, Physical Review Letters **23** (1969), no. 15, 880–884.
- [CHTW04] R. Cleve, P. Hoyer, B. Toner, and J. Watrous, *Consequences and limits of nonlocal strategies*, 19th IEEE Annual Conference on Computational Complexity, 2004, pp. 236–249.
- [CLP17] R. Cleve, L. Liu, and V.I. Paulsen, *Perfect embezzlement of entanglement*, Journal of Mathematical Physics **58** (2017), 012204.
- [CLS17] R. Cleve, L. Liu, and W. Slofstra, *Perfect commuting-operator strategies for linear system games*, Journal of Mathematical Physics **58** (2017), no. 1, 012202.
- [CM14] R. Cleve and R. Mittal, *Characterization of binary constraint system games*, International Colloquium on Automata, Languages, and Programming, 2014, pp. 320–331.
- [CSUU06] R. Cleve, W. Slofstra, F. Unger, and S. Upadhyay, *Perfect parallel repetition theorem for quantum XOR proof systems*, Computational Complexity **17** (2006), no. 2, 282–299.
- [Col20] A. Coladangelo, *A two-player dimension witness based on embezzlement, and an elementary proof of the non-closure of the set of quantum correlations*, Quantum **4** (2020), 282–300.
- [CS17] A. Coladangelo and J. Stark, *Robust self-testing for linear constraint system games*, preprint (arxiv:1709.09267) (2017).
- [CS18] A. Coladangelo and J. Stark, *Unconditional separation of finite and infinite-dimensional quantum correlations*, preprint (arXiv:1804.05116) (2018).
- [Con76] A. Connes, *Classification of Injective Factors, Cases II_1 , II_∞ , III_λ , $\lambda \neq 1$* , Annals of Mathematics (2) **104** (1976), no. 1, 73–115.

- [Con00] J.B. Conway, *A course in operator theory*, Graduate Studies in Mathematics, vol. 21, American Mathematical Society, Providence, RI, 2000.
- [CKLT16] J. Crann, D.W. Kribs, R.H. Levene, and I.G. Todorov, *Private algebras in quantum information and infinite-dimensional complementarity*, Journal of Mathematical Physics **57** (2016), 015208.
- [CM23] E. Culf and A. Mehta, *New approaches to complexity via quantum graphs*, preprint (arXiv:2309.12887) (2023).
- [dlCDN20] G. De las Cuevas, T. Drescher, and T. Netzer, *Quantum magic squares: dilations and their limitations*, J. Math. Phys. **61** (2020).
- [DK17] K.R. Davidson and E.T.A. Kakariadis, *A proof of Boca’s theorem*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **149** (2017), 869–876.
- [Daw24] M. Daws, *Quantum graphs: different perspectives, homomorphisms and quantum homomorphisms*, Communications of the American Mathematical Society **4** (2024), 117–181.
- [DP16] K. Dykema and V.I. Paulsen, *Synchronous correlation matrices and Connes’ embedding conjecture*, Journal of Mathematical Physics **57** (2016), 015214.
- [DPP19] K. Dykema, V.I. Paulsen, and J. Prakash, *Non-closure of the set of quantum correlations via graphs*, Communications in Mathematical Physics **365** (2019), 1125–1142.
- [EPR35] A. Einstein, B. Podolsky, and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review **47** (1935), 777–780.
- [EPK11] A. Einstein, K. Przibram, and K. Klein, *Letters on Wave Mechanics: Correspondence with H. A. Lorentz, Max Planck, and Erwin Schrödinger*, Philosophical Library/Open Road, 2011.
- [FKPT18] D. Farenick, A.S. Kavruk, V.I. Paulsen, and I.G. Todorov, *Characterisations of the weak expectation property*, New York Journal of Mathematics **24a** (2018), 107–135.
- [FK00] U. Feige and J. Kilian, *Two-prover protocols—low error at affordable rates*, SIAM Journal on Computing **30** (2000), no. 1, 324–346.
- [FR87] P. Frankl and V. Rodl, *Forbidden intersections*, Trans. American Mathematical Society **300** (1987), 259–286.
- [Fre22] A. Frei, *Connes implies Tsirelson: a simple proof*, preprint (arXiv:2209.07940) (2022).
- [Fri12] T. Fritz, *Tsirelson’s problem and Kirchberg’s conjecture*, Reviews in Mathematical Physics **24** (2012), 1250012, 67pp.
- [Gol21] A. Goldberg, *Synchronous linear constraint system games*, Journal of Mathematical Physics **62** (2021), 032201.
- [GH23] I. Goldbring and B. Hart, *A survey on the model theory of tracial von Neumann algebras*, Model theory of operator algebras, 2023, DOI doi:10.1515/9783110768282.
- [Gro53] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo **8** (1953), 1–79.
- [HM11] U. Haagerup and M. Musat, *Factorization and dilation problems for completely positive maps on von Neumann algebras*, Communications in Mathematical Physics **303** (2011), 555–594.
- [HM15] ———, *An asymptotic property of factorizable completely positive maps and the Connes embedding problem*, Communications in Mathematical Physics **338** (2015), 721–752.
- [Har17] S.J. Harris, *Connes’ embedding problem and winning strategies for quantum XOR games*, Journal of Mathematical Physics **58** (2017), 122203.

- [Har22] ———, *Synchronous games with $\ast$ -isomorphic game algebras*, Quantum Information and Computation **22** (2022), no. 11 & 12, 924–946.
- [Har24] ———, *Universality of graph homomorphism games and the quantum coloring problem*, Annales Henri Poincaré (2024).
- [HLP⁺18] S.J. Harris, R.H. Levene, V.I. Paulsen, S. Plosker, and M. Rahaman, *Schur multipliers and mixed unitary maps*, Journal of Mathematical Physics **59** (2018).
- [HMPS19] J.W. Helton, K.P. Meyer, V.I. Paulsen, and M. Satriano, *Algebras, synchronous games and chromatic numbers of graphs*, New York Journal of Mathematics **25** (2019), 328–361.
- [HMN⁺24] J.W. Helton, H. Mousavi, S.S. Nezhadi, V.I. Paulsen, and T.B. Russell, *Synchronous values of games*, Annales Henri Poincaré (2024).
- [Hol09] T. Holenstein, *Parallel repetition: simplification and the no-signaling case*, Theory of Computing **5** (2009), no. 8, 141–172.
- [JPY14] R. Jain, A. Pereszlényi, and P. Yao, *A parallel repetition theorem for entangled two-player one-round games under product distributions*, 2014 IEEE 29th Conference on Computational Complexity (CCC), 2014, pp. 209–216.
- [Ji13] Z. Ji, *Binary constraint system games and locally commutative reductions*, preprint (arXiv:1310.3794) (2013).
- [JNV⁺20] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen, $MIP^* = RE$, preprint (arXiv:2001.04383) (2020).
- [JNP⁺11] M. Junge, M. Navascués, C. Palazuelos, D. Perez-Garcia, V.B. Scholz, and R.F. Werner, *Connes’ embedding problem and Tsirelson’s problem*, Journal of Mathematical Physics **52** (2011), 012102, 12pp.
- [KM19] S.J. Kim and A. Mehta, *Chromatic numbers, Sabidussi’s Theorem and Hedetniemi’s conjecture for non-commutative graphs*, Linear Algebra and its Applications **582** (2019), 291–309.
- [KPS18] S.J. Kim, V.I. Paulsen, and C. Schafhauser, *A synchronous game for binary constraint systems*, Journal of Mathematical Physics **59** (2018), 032201.
- [Kir94] E. Kirchberg, *Discrete groups with Kazhdan’s property T and factorization property are residually finite*, Mathematische Annalen **299** (1994), 35–63.
- [LMW⁺22] Z.D. Li, Y.L. Mao, M. Weilenmann, A. Tavakoli, H. Chen, L. Feng, S.J. Yang, M.-O. Renou, D. Trillo, T.P. Le, N. Gisin, A. Acín, M. Navascués, Z. Wang, and J. Fan, *Testing real quantum theory in an optical quantum network*, Physical Review Letters **128** (2022), no. 4, 040402.
- [LMP⁺20] M. Lupini, L. Mančinska, V.I. Paulsen, D.E. Roberson, G. Scarpa, S. Severini, I.G. Todorov, and A. Winter, *Perfect strategies for non-signalling games*, Mathematical Physics, Analysis and Geometry **23** (2020), no. 7.
- [MPTW23] L. Mančinska, V.I. Paulsen, I.G. Todorov, and A. Winter, *Products of synchronous games*, Studia Mathematica **272** (2023), no. 3, 299–317.
- [MR12] L. Mančinska and D.E. Roberson, *Graph homomorphisms for quantum players*, preprint (arXiv:1212.1724) (2012).
- [MR16a] ———, *Quantum homomorphisms*, Journal of Combinatorial Theory, Series B **118** (2016), 228–267.
- [MR16b] ———, *Oddities of quantum colorings*, Baltic Journal of Modern Computing **4** (2016), no. 4, 846–859.
- [MR20] ———, *Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs*, 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS), 2020, pp. 661–672.
- [MRV15] L. Mančinska, D.E. Roberson, and A. Varvitsiotis, *Deciding the existence of perfect entangled strategies for nonlocal games*, preprint (arXiv:1506.07429) (2015).

- [MV18] L. Mančinska and T. Vidick, *Unbounded entanglement in non-local games*, preprint(arxiv:1402.4145) (2018).
- [Mer90] N.D. Mermin, *Simple unified form for the major no-hidden variables theorems*, Physical Review Letters **65** (1990), no. 27, 3373–3376.
- [MR20] M. Musat and M. Rørdam, *Non-closure of quantum correlation matrices and factorizable channels that require infinite dimensional ancilla (with an appendix by Narutaka Ozawa)*, Communications in Mathematical Physics **375** (2020), 1761–1776.
- [NPA08] M. Navascués, S. Pironio, and A. Acín, *A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations*, New Journal of Physics **10** (2008), 073013.
- [Oza13] N. Ozawa, *About the Connes embedding conjecture—algebraic approaches*, Japanese Journal of Mathematics **8** (2013), 147–183.
- [PS23] C. Paddock and W. Slofstra, *Satisfiability problems and algebras of boolean constraint system games*, preprint(arXiv:2310.07901) (2023).
- [PV10] K.F. Páí and T. Vértesi, *Maximal violation of the I3322 inequality using infinite dimensional quantum systems*, Phys. Rev. A **82** (2010).
- [Pau16] V.I. Paulsen, *Entanglement and non-locality*, Online course notes (2016).
- [PR21] V.I. Paulsen and M. Rahaman, *Bisynchronous games and factorizable maps*, Annales Henri Poincaré **22** (2021), 593–614.
- [PSS⁺16] V.I. Paulsen, S. Severini, D. Stahlke, I.G. Todorov, and A. Winter, *Estimating quantum chromatic numbers*, Journal of Functional Analysis **270** (2016), no. 6, 2188–2222.
- [PT15] V.I. Paulsen and I.G. Todorov, *Quantum chromatic numbers via operator systems*, Quarterly Journal of Mathematics **66** (2015), no. 2, 677–692.
- [Raz98] R. Raz, *A parallel repetition theorem*, SIAM Journal on Computing **27** (1998), no. 3, 763–803.
- [RV15] O. Regev and T. Vidick, *Quantum XOR games*, ACM Transactions on Computation Theory **7** (2015), no. 15, 1–43.
- [RTW⁺21] M.O. Renou, D. Trillo, M. Weilenmann, T.P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, *Quantum theory based on real numbers can be experimentally falsified*, Nature **600** (2021), 625–629.
- [Rob13] D.E. Roberson, *Variations on a theme: graph homomorphisms*, Ph.D. Thesis (University of Waterloo), 2013.
- [ST02] Y. Samoilenko and L. Turowska, *On bounded and unbounded idempotents whose sum is a multiple of the identity*, Methods of Functional Analysis and Topology **8** (2002), no. 1, 79–100.
- [Slo19a] W. Slofstra, *The set of quantum correlations is not closed*, Forum of Mathematics, Pi **7** (2019), e1.
- [Slo19b] ———, *Tsirelson’s problem and an embedding theorem for groups arising from non-local games*, Journal of the American Mathematical Society **33** (2019), no. 1, 1–56.
- [Sta16] D. Stahlke, *Quantum zero-error source-channel coding and non-commutative graph theory*, IEEE Transactions on Information Theory **62** (2016), no. 1, 554–577.
- [TT24] I.G. Todorov and L. Turowska, *Quantum no-signalling correlations and non-local games*, Communications in Mathematical Physics **405** (2024).
- [Tsi80] B.S. Tsirelson, *Quantum generalizations of Bell’s inequality*, Letters in Mathematical Physics **4** (1980), 93–100.
- [Tsi93] ———, *Some results and problems on quantum Bell-type inequalities*, Hadronic Journal Supplement **8** (1993), no. 4, 329–345.
- [vDH03] W. van Dam and P. Hayden, *Universal entanglement transformations without communication*, Physical Review A **67** (2003), 060302.

- [Vid22] T. Vidick, *Almost synchronous quantum correlations*, Journal of Mathematical Physics **63** (2022), 022201.
- [VW15] T. Vidick and J. Watrous, *Quantum Proofs*, Foundations and Trends in Theoretical Computer Science **11** (2015), 1-215.
- [Wea21] N. Weaver, *Quantum graphs as quantum relations*, Journal of Geometric Analysis **31** (2021), 9090–9112.
- [Yue16] H. Yuen, *A parallel repetition theorem for all entangled games*, International Colloquium of Automata, Languages, and Programming (ICALP 2016), 2016, pp. 77:1–77:13.

DEPARTMENT OF MATHEMATICS & STATISTICS, NORTHERN ARIZONA UNIVERSITY, 801
S. OSBORNE DRIVE, FLAGSTAFF, AZ, USA, 86001
Email address: `samuel.harris@nau.edu`

INSTITUTE FOR QUANTUM COMPUTING AND DEPARTMENT OF PURE MATHEMATICS,
UNIVERSITY OF WATERLOO, WATERLOO, ON, CANADA N2L 3G1
Email address: `vpaulsen@uwaterloo.ca`