Exercises for Quantum Information
Sheet 2 — Postulates of Quantum Mechanics

Recall the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Exercise 1. Let M be an observable and $|\varphi\rangle$ be a state. Show that the expected value $\mathbb{E}(M)$ on $|\varphi\rangle$ can indeed be computed as $\langle\varphi|M|\varphi\rangle$.

Exercise 2. Decide which of the above matrices X, Y, Z, H represents observables. For those which do, compute their expected measurement value on states $|0\rangle$, $|1\rangle$, $|+\rangle$ and $|-\rangle$.

Exercise 3. Show that the operator $P_{|\varphi\rangle} = |\varphi\rangle\langle\varphi|$ is indeed the projection operator onto the linear space generated by $|\varphi\rangle$.

Exercise 4. Let $|\phi\rangle$ be first measured in basis $\{|0\rangle, |1\rangle\}$, then in basis $\{|+\rangle, |-\rangle\}$ and again in basis $\{|0\rangle, |1\rangle\}$. Compute the probability of the last measurement being equal to $|1\rangle$. Compute the conditional probability of the last measurement being equal to $|1\rangle$ assuming the first measurement equals $|1\rangle$. How do these numbers change if we drop the middle measurement and leave only the first and the last measurements?

Exercise 5. Show that the average value of the observable $X_1 Z_2$ for a two qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.