Exercises for Quantum Information
Sheet 2 — Postulates of Quantum Mechanics

Recall the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Exercise 1. Let M be an observable and $|\varphi\rangle$ be a state. Show that the expected value $\mathbb{E}(M)$ on $|\varphi\rangle$ can indeed be computed as $\langle\varphi|M|\varphi\rangle$.

Solution. Let M be an observable with spectral decomposition $M = \sum_m m\, P_m$, where P_m are the projectors onto the eigenspaces of M. The probability of obtaining outcome m in state $|\psi\rangle$ is $p(m) = \langle\psi|P_m|\psi\rangle$. Hence the expectation value is

$$\mathbb{E}(M) = \sum_m m\, p(m) = \sum_m m\, \langle\psi|P_m|\psi\rangle = \langle\psi|\Big(\sum_m m\, P_m\Big)|\psi\rangle = \langle\psi|M|\psi\rangle.$$

□

Exercise 2. Decide which of the above matrices X, Y, Z, H represents observables. For those which do, compute their expected measurement value on states $|0\rangle$, $|1\rangle$, $|+\rangle$ and $|-\rangle$.

Solution. An observable must be Hermitian. The matrices X, Y, Z are Hermitian, and H is real symmetric, hence also Hermitian. So all four are observables. Their expectation values are $\langle\psi|M|\psi\rangle$:

state $\lvert\psi\rangle$	$\langle X\rangle_\psi$	$\langle Y\rangle_\psi$	$\langle Z\rangle_\psi$	$\langle H\rangle_\psi$
$\lvert 0\rangle$	0	0	1	$1/\sqrt{2}$
$\lvert 1\rangle$	0	0	-1	$-1/\sqrt{2}$
$\lvert +\rangle$	1	0	0	$1/\sqrt{2}$
$\lvert -\rangle$	-1	0	0	$1/\sqrt{2}$

For example, $X|+\rangle = |+\rangle$ so $\langle+|X|+\rangle = 1$, and $Z|0\rangle = |0\rangle$ so $\langle 0|Z|0\rangle = 1$. □

Exercise 3. Show that the operator $P_{|\varphi\rangle} = |\varphi\rangle\langle\varphi|$ is indeed the projection operator onto the linear space generated by $|\varphi\rangle$.

Solution. Assume $|\varphi\rangle$ is normalized. For any vector $|\psi\rangle$,

$$P_\varphi|\psi\rangle = |\varphi\rangle\langle\varphi|\psi\rangle.$$

So $P_\varphi|\psi\rangle$ is always a scalar multiple of $|\varphi\rangle$, hence lies in $\operatorname{span}\{|\varphi\rangle\}$.

Also, if $|\psi\rangle = a|\varphi\rangle$ already lies in this subspace, then

$$P_\varphi|\psi\rangle = |\varphi\rangle\langle\varphi|a\varphi\rangle = a|\varphi\rangle = |\psi\rangle.$$

And if $|\psi\rangle$ is orthogonal to $|\varphi\rangle$, then $\langle\varphi|\psi\rangle = 0$, so $P_\varphi|\psi\rangle = 0$.

Thus P_φ acts as the identity on $\operatorname{span}\{|\varphi\rangle\}$ and kills its orthogonal complement, so it is exactly the projector onto the line generated by $|\varphi\rangle$. □

Exercise 4. Let $|\phi\rangle$ be first measured in basis $\{|0\rangle, |1\rangle\}$, then in basis $\{|+\rangle, |-\rangle\}$ and again in basis $\{|0\rangle, |1\rangle\}$. Compute the probability of the last measurement being equal to $|1\rangle$. Compute the conditional probability of the last measurement being equal to $|1\rangle$ assuming the first measurement equals $|1\rangle$. How do these numbers change if we drop the middle measurement and leave only the first and the last measurements?

Solution. Write $|\phi\rangle = a|0\rangle + b|1\rangle$, so $|a|^2 + |b|^2 = 1$.

After the first measurement in the $\{|0\rangle, |1\rangle\}$ basis, the state becomes $|0\rangle$ with probability $|a|^2$ and $|1\rangle$ with probability $|b|^2$.

Now consider the second measurement in the $\{|+\rangle, |-\rangle\}$ basis. Using

$$|0\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \qquad |1\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle),$$

we see that measuring $|0\rangle$ or $|1\rangle$ in this basis yields $|+\rangle$ or $|-\rangle$ with probability $1/2$ each. Thus after the second measurement the state is either $|+\rangle$ or $|-\rangle$.

Finally, measuring in the $\{|0\rangle, |1\rangle\}$ basis again, we use

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

so from either $|+\rangle$ or $|-\rangle$ the probability of obtaining $|1\rangle$ is again $1/2$. Hence regardless of the outcome of the first two measurements, the probability that the last measurement is $|1\rangle$ is $1/2$.

Therefore

$$\Pr(\text{last } = |1\rangle) = \tfrac{1}{2}, \qquad \Pr(\text{last } = |1\rangle \mid \text{first } = |1\rangle) = \tfrac{1}{2}.$$

If we drop the middle measurement, then after the first measurement the state is already either $|0\rangle$ or $|1\rangle$, and measuring again in the same basis gives the same outcome with certainty. Thus

$$\Pr(\text{last } = |1\rangle) = |b|^2, \qquad \Pr(\text{last } = |1\rangle \mid \text{first } = |1\rangle) = 1.$$

□

Exercise 5. Show that the average value of the observable X_1Z_2 for a two qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.

Solution. Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Here $X_1 Z_2$ means $X \otimes Z$. Now

$$(X \otimes Z)|00\rangle = |1\rangle \otimes |0\rangle = |10\rangle, \qquad (X \otimes Z)|11\rangle = |0\rangle \otimes (-|1\rangle) = -|01\rangle.$$

So

$$(X \otimes Z)|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}.$$

Therefore

$$\langle\psi|(X \otimes Z)|\psi\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|10\rangle - |01\rangle) = 0,$$

since the computational basis states are orthonormal. Hence the average value of $X_1 Z_2$ in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is 0. □