Exercises for Quantum Information
Sheet 1 — Linear Algebra

1 Matrices, orthogonality, eigenvalues, eigenvectors

Exercise 1 (Matrix representations: example)**.** Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Exercise 2 (Matrix representation for operator products)**.** Suppose A is a linear operator from vector space V to vector space W, and B is a linear operator from vector space W to vector space X. Let $\{|v_i\rangle\}$, $\{|w_j\rangle\}$, and $\{|x_k\rangle\}$ be bases for the vector spaces V, W, and X, respectively. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to the appropriate bases.

Exercise 3 (Basis changes)**.** Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases, $\{|v_i\rangle\}$ and $\{|w_i\rangle\}$. Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

Exercise 4. (1) Compute the norm of the complex vector (0) in the one-dimensional arithmetic vector space over $\mathbb{C}$. Compute the norm of the complex vector $|0\rangle \in \mathbb{H}_2$.

(2) Show that $|+\rangle$, $|-\rangle$ forms an orthonormal basis of $\mathbb{H}_2$.

(3) Express $|0\rangle$ and $|1\rangle$ as linear combinations of $|+\rangle$ and $|-\rangle$. Then, compute the corresponding transition matrix.

Exercise 5. Suppose $\{|v_i\rangle\}$ is an orthonormal basis for an inner product space V. What is the matrix representation for the operator $|v_j\rangle\langle v_k|$, with respect to the $|v_i\rangle$ basis?

Exercise 6. If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Exercise 7. Compute the inverses of X, Y, Z, H. Which of these matrices are normal?

Exercise 8 (Eigendecomposition of the Pauli matrices)**.** Find the eigenvectors, eigenvalues, and diagonal form, and spectral decomposition of the Pauli matrices X, Y, and Z, and the 2×2 Hadamard matrix H.

Exercise 9. Prove that the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is not diagonalizable.

Exercise 10. If M is an operator on $\mathbb{C}^2$, how can we compute the eigenvalues of M from $\det M$ and $\operatorname{tr} M$?

Exercise 11. Consider the matrix $A = |0\rangle\langle 0| + |+\rangle\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Compute its eigenvectors and eigenvalues.

Exercise 12. Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

Exercise 13. Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Exercise 14. Show that the Pauli matrices are Hermitian and unitary.

Exercise 15. Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

Exercise 16 (Hermiticity of positive operators)**.** Show that a positive semidefinite operator is necessarily Hermitian. *Hint:* Show that an arbitrary operator A can be written $A = B + iC$ where B and C are Hermitian.

Exercise 17. Show that for every operator A, $A^\dagger A$ is positive semidefinite.

2 Tensor products

Exercise 18. Compute the dot product of vectors $|2\rangle$ and $|3\rangle$ from $\mathbb{H}_{16} \cong \mathbb{H}_2^{\otimes^4}$.

Exercise 19. Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product.

Exercise 20. Calculate the matrix representation of the tensor products of the Pauli operators (a) $X \otimes Z$; (b) $I \otimes X$; (c) $X \otimes I$. Is the tensor product commutative?

Exercise 21. Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,

$$(A \otimes B)^* = A^* \otimes B^*, \qquad (A \otimes B)^T = A^T \otimes B^T, \qquad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger.$$

Exercise 22. Show that the tensor product of two $\{\text{unitary}, \text{Hermitian}, \text{positive}, \text{projection}\}$ operators is a $\{\text{unitary}, \text{Hermitian}, \text{positive}, \text{projection}\}$ operator.

Exercise 23. Let H_2 be $H \otimes H$. Compute H_2 and the write it is a linear combination of projection operators (*spectral decomposition*).