Exercises for Quantum Information
Sheet 1 — Linear Algebra

1 Matrices, orthogonality, eigenvalues, eigenvectors

Exercise 1 (Matrix representations: example)**.** Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Solution. In the basis $\{|0\rangle, |1\rangle\}$, the columns of A are $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$, so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we instead use the basis $\{|+\rangle, |-\rangle\}$ with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$, then $A|+\rangle = |+\rangle$ and $A|-\rangle = -|-\rangle$. Thus the matrix becomes $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, a different representation of the same operator. □

Exercise 2 (Matrix representation for operator products)**.** Suppose A is a linear operator from vector space V to vector space W, and B is a linear operator from vector space W to vector space X. Let $\{|v_i\rangle\}$, $\{|w_j\rangle\}$, and $\{|x_k\rangle\}$ be bases for the vector spaces V, W, and X, respectively. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to the appropriate bases.

Solution. Let $A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$ and $B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$ be the matrix representations of A and B in the given bases. Then

$$BA|v_i\rangle = B\left(\sum_j A_{ji}|w_j\rangle\right) = \sum_j A_{ji}B|w_j\rangle = \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle = \sum_k\Big(\sum_j B_{kj}A_{ji}\Big)|x_k\rangle.$$

Thus the matrix elements of BA in the bases $\{|v_i\rangle\}$ and $\{|x_k\rangle\}$ are $(BA)_{ki} = \sum_j B_{kj}A_{ji}$, which is exactly the matrix product of B and A. □

Exercise 3 (Basis changes)**.** Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases, $\{|v_i\rangle\}$ and $\{|w_i\rangle\}$. Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

Solution. Let U be the change-of-basis unitary with $U_{ij} = \langle v_i | w_j \rangle$. Then $|w_j\rangle = \sum_k U_{kj} |v_k\rangle$, and a short calculation gives

$$A'' = U^\dagger A' U.$$

Thus matrix representations in different orthonormal bases are related by a unitary similarity transform. □

Exercise 4. (1) Compute the norm of the complex vector (0) in the one-dimensional arithmetic vector space over $\mathbb{C}$. Compute the norm of the complex vector $|0\rangle \in \mathbb{H}_2$.

(2) Show that $|+\rangle$, $|-\rangle$ forms an orthonormal basis of $\mathbb{H}_2$.

(3) Express $|0\rangle$ and $|1\rangle$ as linear combinations of $|+\rangle$ and $|-\rangle$. Then, compute the corresponding transition matrix.

Solution. (1) In a 1D complex vector space, the zero vector has norm 0. In $\mathbb{H}_2$, the computational basis is orthonormal, so $\| |0\rangle \| = 1$.

(2) By definition $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Then $\langle + | + \rangle = 1$, $\langle - | - \rangle = 1$, and $\langle + | - \rangle = 0$, so $\{|+\rangle, |-\rangle\}$ is an orthonormal basis.

(3) Solve for $|0\rangle, |1\rangle$:

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

gives

$$|0\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |1\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Thus the transition matrix between the bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

□

Exercise 5. Suppose $\{|v_i\rangle\}$ is an orthonormal basis for an inner product space V. What is the matrix representation for the operator $|v_j\rangle\langle v_k|$, with respect to the $|v_i\rangle$ basis?

Solution. The operator sends $|v_k\rangle$ to $|v_j\rangle$ and annihilates all other basis vectors. Thus its matrix has a single 1 in row j, column k, and zeros elsewhere: $(|v_j\rangle\langle v_k|)_{ab} = \delta_{aj}\,\delta_{bk}$. □

Exercise 6. If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Solution. For any vectors $|a\rangle, |b\rangle$,

$$\langle a | (|w\rangle\langle v|)^\dagger | b\rangle = \big(\langle b || w\rangle\langle v || a\rangle\big)^* = \big(\langle b | w\rangle \langle v | a\rangle\big)^* = \langle a | v\rangle \langle w | b\rangle = \langle a || v\rangle\langle w || b\rangle.$$

Since the matrix elements agree on all $|a\rangle, |b\rangle$, the operators are equal. □

Exercise 7. Compute the inverses of X, Y, Z, H. Which of these matrices are normal?

Solution. All Pauli matrices satisfy $X^2 = Y^2 = Z^2 = I$, so each is its own inverse:

$$X^{-1} = X, \qquad Y^{-1} = Y, \qquad Z^{-1} = Z.$$

For the Hadamard operator H, we also have $H^2 = I$, hence

$$H^{-1} = H.$$

□

Exercise 8 (Eigendecomposition of the Pauli matrices)**.** Find the eigenvectors, eigenvalues, and diagonal form, and spectral decomposition of the Pauli matrices X, Y, and Z, and the 2×2 Hadamard matrix H.

Solution. For each matrix, eigenvalues are ± 1, so the diagonal form in its eigenbasis is always $\mathrm{diag}(1, -1)$.

Z: $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$, so eigenpairs are $|0\rangle \leftrightarrow 1$, $|1\rangle \leftrightarrow -1$. Diagonal form: $\mathrm{diag}(1, -1)$ in $\{|0\rangle, |1\rangle\}$. Spectral decomposition: $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

X: eigenvectors $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ with $X|+\rangle = |+\rangle$, $X|-\rangle = -|-\rangle$. Diagonal form: $\mathrm{diag}(1, -1)$ in $\{|+\rangle, |-\rangle\}$. Spectral decomposition: $X = |+\rangle\langle +| - |-\rangle\langle -|$.

Y: eigenvectors $|y_+\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$, $|y_-\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$ with $Y|y_\pm\rangle = \pm|y_\pm\rangle$. Diagonal form: $\mathrm{diag}(1, -1)$ in $\{|y_+\rangle, |y_-\rangle\}$. Spectral decomposition: $Y = |y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|$.

H: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ satisfies $H^2 = I$, so eigenvalues are ± 1. Solving $H(a|0\rangle + b|1\rangle) = \lambda(a|0\rangle + b|1\rangle)$ gives eigenvectors proportional to $(1+\sqrt{2})|0\rangle + |1\rangle$ for $\lambda = 1$ and $(1-\sqrt{2})|0\rangle + |1\rangle$ for $\lambda = -1$ (normalize to get $|h_+\rangle, |h_-\rangle$). Diagonal form: $\mathrm{diag}(1, -1)$ in $\{|h_+\rangle, |h_-\rangle\}$. Spectral decomposition: $H = |h_+\rangle\langle h_+| - |h_-\rangle\langle h_-|$. □

Exercise 9. Prove that the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is not diagonalizable.

Solution. Its characteristic polynomial is $(1 - \lambda)^2$, so the only eigenvalue is $\lambda = 1$. Solving $(A - I)|x\rangle = 0$ gives eigenvectors proportional to $(0, 1)$, so the eigenspace is one-dimensional. Since a 2×2 matrix needs two linearly independent eigenvectors to be diagonalizable and this one has only one, it is not diagonalizable. □

Exercise 10. If M is an operator on $\mathbb{C}^2$, how can we compute the eigenvalues of M from $\det M$ and $\operatorname{tr} M$?

Solution. For a 2×2 matrix, the characteristic polynomial is $\lambda^2 - (\operatorname{tr} M)\lambda + \det M = 0$. Thus the eigenvalues are

$$\lambda_\pm = \frac{\operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4 \det M}}{2}.$$

□

Exercise 11. Consider the matrix $A = |0\rangle\langle 0| + |+\rangle\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Compute its eigenvectors and eigenvalues.

Solution. We can compute this via characteristic polynomials, or notice that since $A = |0\rangle\langle 0| + |+\rangle\langle +|$ is a sum of two rank-1 projectors. We can observe that $|0\rangle\langle 0|$ has eigenvalues $\{1, 0\}$ with eigenvectors $|0\rangle, |1\rangle$, and $|+\rangle\langle +|$ has eigenvalues $\{1, 0\}$ with eigenvectors $|+\rangle, |-\rangle$.

Note that $|0\rangle$ is not orthogonal to $|+\rangle$: their overlap is $\langle 0|+\rangle = 1/\sqrt{2}$. Thus the operator acts as

$$A|0\rangle = |0\rangle + \tfrac{1}{2}|0\rangle = \tfrac{3}{2}|0\rangle, \qquad A|1\rangle = 0 \cdot |1\rangle + \tfrac{1}{2}|1\rangle = \tfrac{1}{2}|1\rangle.$$

So $|0\rangle$ and $|1\rangle$ are already eigenvectors with eigenvalues $3/2$ and $1/2$, respectively.

Therefore the eigenvalues are $3/2$ and $1/2$, with corresponding eigenvectors $|0\rangle$ and $|1\rangle$. □

Exercise 12. Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

Solution. If a normal matrix A is Hermitian, then $A = A^\dagger$, so every eigenvalue λ satisfies $\lambda = \lambda^*$ (because $\langle\psi|A|\psi\rangle = \lambda$ is real for any normalized eigenvector $|\psi\rangle$). Hence all eigenvalues are real.

Conversely, if A is normal and all eigenvalues are real, then A is unitarily diagonalizable: $A = UDU^\dagger$ with D real diagonal. Since $D = D^\dagger$, we get $A^\dagger = (UDU^\dagger)^\dagger = UD^\dagger U^\dagger = UDU^\dagger = A$, so A is Hermitian. □

Exercise 13. Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Solution. Let U be unitary and $U|\psi\rangle = \lambda|\psi\rangle$ for some eigenvector $|\psi\rangle$. Then

$$\||\psi\rangle\|^2 = \|U|\psi\rangle\|^2 = \|\lambda|\psi\rangle\|^2 = |\lambda|^2 \, \||\psi\rangle\|^2.$$

Since $\||\psi\rangle\| \neq 0$, we must have $|\lambda|^2 = 1$, so $|\lambda| = 1$. Any complex number of modulus 1 can be written as $e^{i\theta}$ for a real θ. □

Exercise 14. Show that the Pauli matrices are Hermitian and unitary.

Solution. Each Pauli matrix equals its own conjugate transpose, so they are Hermitian:

$$X^\dagger = X, \quad Y^\dagger = Y, \quad Z^\dagger = Z.$$

They are also unitary because $X^2 = Y^2 = Z^2 = I$. Thus $X^\dagger X = Y^\dagger Y = Z^\dagger Z = I$, which is the definition of unitarity. □

Exercise 15. Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

Solution. Let A be Hermitian and let $A|v\rangle = \lambda|v\rangle$ and $A|w\rangle = \mu|w\rangle$ with $\lambda \neq \mu$. Then

$$\lambda \langle v|w\rangle = \langle v|A|w\rangle = \langle v|A^\dagger|w\rangle = \langle v|A|w\rangle = \mu \langle v|w\rangle.$$

Thus $(\lambda - \mu)\langle v|w\rangle = 0$, and since $\lambda \neq \mu$, we get $\langle v|w\rangle = 0$. □

Exercise 16 (Hermiticity of positive operators)**.** Show that a positive semidefinite operator is necessarily Hermitian. *Hint:* Show that an arbitrary operator A can be written $A = B + iC$ where B and C are Hermitian.

Solution. Write any operator as $A = B + iC$ with $B = \frac{1}{2}(A + A^\dagger)$ and $C = \frac{1}{2i}(A - A^\dagger)$, so B, C are Hermitian.

If A is positive semidefinite, then $\langle\psi|A|\psi\rangle \geq 0$ for all $|\psi\rangle$, so these expectation values are real. But

$$\langle\psi|A|\psi\rangle = \langle\psi|B|\psi\rangle + i\,\langle\psi|C|\psi\rangle,$$

with $\langle\psi|B|\psi\rangle, \langle\psi|C|\psi\rangle$ real. For this to be real for all $|\psi\rangle$, we must have $\langle\psi|C|\psi\rangle = 0$ for all $|\psi\rangle$, hence $C = 0$. Thus $A = B = A^\dagger$, so A is Hermitian. □

Exercise 17. Show that for every operator A, $A^\dagger A$ is positive semidefinite.

Solution. For any vector $|\psi\rangle$,

$$\langle\psi|A^\dagger A|\psi\rangle = (A|\psi\rangle)^\dagger(A|\psi\rangle) = \|A|\psi\rangle\|^2 \geq 0.$$

□

2 Tensor products

Exercise 18. Compute the dot product of vectors $|2\rangle$ and $|3\rangle$ from $\mathbb{H}_{16} \cong \mathbb{H}_2^{\otimes 4}$.

Exercise 19. Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product.

Solution. We have $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

For two copies,

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

As a Kronecker product: $|\psi\rangle \otimes |\psi\rangle = \frac{1}{2}(1,1) \otimes (1,1) = \frac{1}{2}(1,1,1,1)$.

For three copies,

$$|\psi\rangle^{\otimes 3} = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle).$$

As a Kronecker product: $(1/\sqrt{2})(1,1)^{\otimes 3} = \frac{1}{\sqrt{8}}(1,1,1,1,1,1,1,1)$. □

Exercise 20. Calculate the matrix representation of the tensor products of the Pauli operators (a) $X \otimes Z$; (b) $I \otimes X$; (c) $X \otimes I$. Is the tensor product commutative?

Solution. In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) $X \otimes Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$

(b) $I \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$

(c) $X \otimes I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$

Since $I \otimes X \neq X \otimes I$, the tensor product is not commutative in general. □

Exercise 21. Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,

$$(A \otimes B)^* = A^* \otimes B^*, \qquad (A \otimes B)^T = A^T \otimes B^T, \qquad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger.$$

Solution. For matrices, $(A \otimes B)_{ij,kl} = A_{ik}B_{jl}$. Taking entrywise complex conjugation gives

$$(A \otimes B)^*_{ij,kl} = (A_{ik}B_{jl})^* = A^*_{ik}\, B^*_{jl} = (A^* \otimes B^*)_{ij,kl}.$$

Similarly, transpose swaps indices:

$$(A \otimes B)^T_{ij,kl} = (A \otimes B)_{kl,ij} = A_{ki}B_{lj} = (A^T \otimes B^T)_{ij,kl}.$$

Finally, the adjoint is conjugate transpose, so combining the two results:

$$(A \otimes B)^\dagger = \big((A \otimes B)^T\big)^* = A^\dagger \otimes B^\dagger.$$

□

Exercise 22. Show that the tensor product of two $\{\text{unitary}, \text{Hermitian}, \text{positive}, \text{projection}\}$ operators is a $\{\text{unitary}, \text{Hermitian}, \text{positive}, \text{projection}\}$ operator.

Solution. Let A and B be operators.

Unitary: if $A^\dagger A = I$ and $B^\dagger B = I$, then

$$(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B) = (A^\dagger A) \otimes (B^\dagger B) = I \otimes I = I,$$

so $A \otimes B$ is unitary.

Hermitian: if $A^\dagger = A$ and $B^\dagger = B$, then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B,$$

so $A \otimes B$ is Hermitian.

PSD: if A, B are PSD, write $A = C^\dagger C$, $B = D^\dagger D$. Then

$$A \otimes B = (C^\dagger C) \otimes (D^\dagger D) = (C^\dagger \otimes D^\dagger)(C \otimes D) = (C \otimes D)^\dagger (C \otimes D),$$

so $A \otimes B$ is positive semidefinite.

Projection: if $A^2 = A$ and $B^2 = B$, then

$$(A \otimes B)^2 = A^2 \otimes B^2 = A \otimes B,$$

so $A \otimes B$ is a projector. □

Exercise 23. Let H_2 be $H \otimes H$. Compute H_2 and the write it is a linear combination of projection operators (*spectral decomposition*).

Solution. Recall $H|+\rangle = |+\rangle$ and $H|-\rangle = -|-\rangle$. Thus for $H_2 = H \otimes H$ we have

$$H_2 |++\rangle = (+1)(+1)|++\rangle = |++\rangle, \quad H_2 |--\rangle = (-1)(-1)|--\rangle = |--\rangle,$$

$$H_2 |+-\rangle = (+1)(-1)|+-\rangle = -|+-\rangle, \quad H_2 |-+\rangle = (-1)(+1)|-+\rangle = -|-+\rangle.$$

So the eigenvalue $+1$ corresponds to eigenvectors $|++\rangle, |--\rangle$ and -1 to $|+-\rangle, |-+\rangle$.

The spectral decomposition is

$$H_2 = |++\rangle\langle++| + |--\rangle\langle--| - |+-\rangle\langle+-| - |-+\rangle\langle-+|.$$

□