Permutation groups – homework

Problem 1. Recall that a simple graph is a tuple (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$. A digraph is a tuple (V, E), where $E \subseteq V^2$.

(1) Construct a simple graph X such that $\mathrm{Aut}(X)$ is isomorphic to the cyclic group $\mathbb{Z}_n$.

(2) Let $G = \{g_1, \ldots, g_n\}$ be a group. Define a colored digraph X with $V(X) = G$ such that there is a directed edge of color k from g_i to g_j if

$$g_i g_j^{-1} = g_k \iff g_i = g_k g_j.$$

What is $\mathrm{Aut}(X)$? Note that an automorphism of a colored digraph has to preserve the colors and also the orientations of the edges.

(3) (Frucht's theorem) Given a finite group G, there exists a simple graph X with $\mathrm{Aut}(X) \cong G$.