

Mathematical Analysis 2

Metric spaces 2

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A. Open and closed sets

What is a neighbourhood? What are open and closed sets? What is a ball of radius r ?

1 Unit balls

What does the unit ball, that is, the set

$$B(a, 1) = \{x \in \mathbb{R}^n : d(a, x) < 1\},$$

look like in $\mathbb{R}^n$ with each of the following metrics?

- the Euclidean metric;
- the Manhattan (sum) metric;
- the maximum metric;
- the discrete metric.

Recall that on any nonempty set M , the discrete metric is defined by

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

2 Unions and intersections

Let (M, ρ) be a metric space. Justify the following statements.

- (a) M and $\emptyset$ are both open and closed in M .
- (b) The union of any family of open sets is open.
- (c) The intersection of finitely many open sets is open.
- (d) The analogous statements hold for closed sets: a finite union of closed sets is closed, and an arbitrary intersection of closed sets is closed.

3 Balls, singletons, and isolated points

Are the following statements true? Justify your answers.

- (a) For $r > 0$, the set $B(a, r) = \{x \in M : d(a, x) < r\}$ is open in the metric space (M, d) .
- (b) For $r > 0$, the set $B'(a, r) = \{x \in M : d(a, x) \leq r\}$ is closed in the metric space (M, d) .
- (c) Every singleton is closed in a metric space (M, d) .
- (d) Describe the open sets in a metric space (M, d) in which every point is isolated.

A point $a \in A \subseteq M$ is an *isolated point* of A if there exists $r > 0$ such that $B(a, r) \cap A = \{a\}$.

B. Continuity of real-valued functions of several variables

How is a continuous mapping defined? Which results about continuous mappings do you know? Consider composition, the characterization by preimages, and the characterization by convergent sequences.

4 Continuity of polynomials

Show that polynomials in several variables are continuous functions $P : \mathbb{R}^n \rightarrow \mathbb{R}$. For simplicity, you may restrict attention to polynomials in two variables.

5 Domains, continuity, level sets, and extrema

Investigate the following functions $f, g, h : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, where $\mathbb{R}^2$ is equipped, for example, with the Euclidean metric:

$$f(x, y) = \sqrt{\ln(x - y)},$$

$$g(x, y) = \sqrt{\frac{1}{x^2} - \frac{y^2}{x^2} - 1},$$

$$h(x, y) = \log(\sqrt{y + 1} - x).$$

For each function, determine its domain. Is the domain open or closed? Investigate continuity and describe the level sets. Does the function attain a global maximum or a global minimum on its domain?

Solutions

Lecture references below refer to Martin Klazar's Lecture 1 (1 October 2026) and Lecture 2 (8 October 2026), with their original numbering.

Terminology: balls, neighbourhoods, and open and closed sets

Let (M, d) be a metric space. For $a \in M$ and $r > 0$, the **open ball** and the **closed ball** are, respectively,

$$B(a, r) = \{x \in M : d(a, x) < r\}, \quad B'(a, r) = \{x \in M : d(a, x) \leq r\}.$$

A **neighbourhood** of a is a set containing some $B(a, r)$ with $r > 0$. A set $U \subseteq M$ is **open** if, for every $a \in U$, there exists $r > 0$ such that $B(a, r) \subseteq U$. A set $F \subseteq M$ is **closed** if $M \setminus F$ is open. The ball and open/closed-set definitions agree with Lecture 1, pp. 6–7 (Definition 14). All these notions refer to the specified ambient space M and its metric.

1. Unit balls

Write $a = (a_1, \dots, a_n)$. In each case we substitute the formula for the metric into $d(a, x) < 1$. Points at distance exactly 1 are excluded.

- **Euclidean (L^2) metric.**

$$B_{d_2}(a, 1) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n (x_i - a_i)^2 < 1 \right\}.$$

This is the usual open ball of radius 1 centred at a : an open disk in $\mathbb{R}^2$ and the interior of a sphere in $\mathbb{R}^3$.

- **Manhattan (sum, L^1) metric.**

$$B_{d_1}(a, 1) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i - a_i| < 1 \right\}.$$

In $\mathbb{R}^2$ this is the interior of a diamond with vertices $a \pm (1, 0)$ and $a \pm (0, 1)$. In $\mathbb{R}^3$ it is the interior of an octahedron. In general, it is the interior of the convex hull of the $2n$ points $a \pm e_i$, where e_i are the standard basis vectors.

- **Maximum (L^∞) metric.**

$$B_{d_{\max}}(a, 1) = \{x \in \mathbb{R}^n : |x_i - a_i| < 1 \text{ for every } i\} = \prod_{i=1}^n (a_i - 1, a_i + 1).$$

This is an open cube of side length 2 with sides parallel to the coordinate axes; in $\mathbb{R}^2$ it is the interior of a square.

- **Discrete metric.** The centre a has distance 0 from itself, whereas every other point has distance 1. Hence

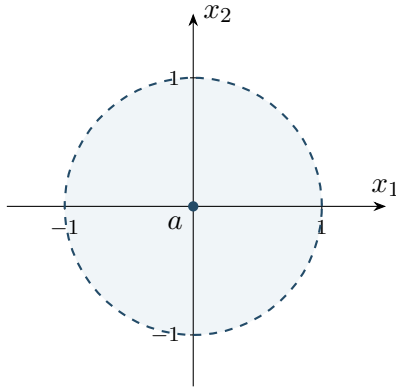
$$\boxed{B_d(a, 1) = \{a\}.$$

More generally, $B_d(a, r) = \{a\}$ for $0 < r \leq 1$, and $B_d(a, r) = M$ for $r > 1$. This holds on any nonempty set M .

1. Unit balls in the plane: illustrations

The pictures show $B_d(a, 1)$ in $\mathbb{R}^2$ with centre $a = (0, 0)$. For any other centre, translate the picture by a . Shaded interiors and filled points are included; dashed outlines are excluded because the defining inequality is $d(a, x) < 1$.

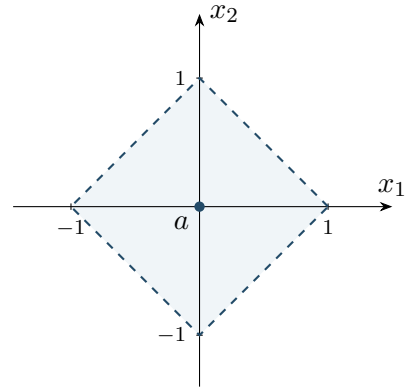
Euclidean metric



$$B_{d_2}(0, 1) = \{x : x_1^2 + x_2^2 < 1\}.$$

An open disk; the circle is excluded.

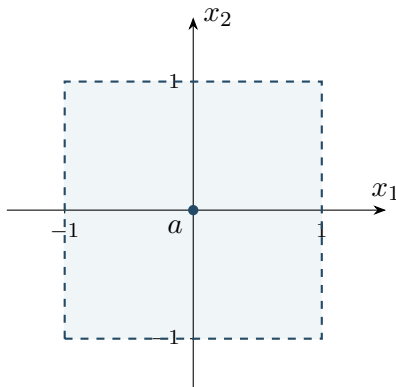
Manhattan (sum) metric



$$B_{d_1}(0, 1) = \{x : |x_1| + |x_2| < 1\}.$$

An open diamond; its edges and vertices are excluded.

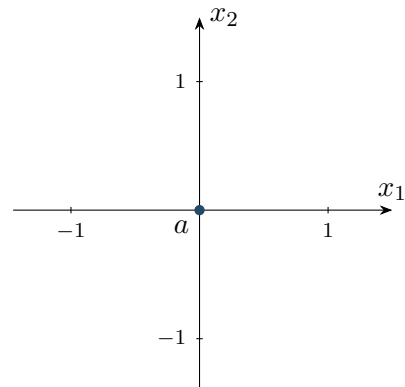
Maximum metric



$$B_{d_{\max}}(0, 1) = (-1, 1) \times (-1, 1).$$

An open square; its edges and vertices are excluded.

Discrete metric



$$B_d(0, 1) = \{(0, 0)\}.$$

Only the centre is included. Every other point has discrete distance 1 from a .

The first three pictures use the same coordinate scale, so they also show the inclusions

$$B_{d_1}(0, 1) \subseteq B_{d_2}(0, 1) \subseteq B_{d_{\max}}(0, 1).$$

The singleton in the fourth picture is open in the discrete metric, even though it is not open in the Euclidean metric.

2. Unions and intersections

These are Proposition 16 and Corollary 20 in Lecture 1, together with the finite-family versions in Exercises 18 and 21.

- (a) **The whole space and the empty set.** For every $a \in M$ and every $r > 0$, we have $B_\rho(a, r) \subseteq M$, so M is open. The definition of openness is automatically satisfied by $\emptyset$, because it has no points at which the condition could fail. Thus $\emptyset$ is open as well. Since their complements in M are each other, both sets are also closed. A set that is both open and closed is called **clopen**.
- (b) **Arbitrary unions of open sets.** Let $(U_i)_{i \in I}$ be any family of open subsets of M , and put $U = \bigcup_{i \in I} U_i$. If $a \in U$, then $a \in U_j$ for some $j \in I$. Openness of U_j gives $r > 0$ such that

$$B_\rho(a, r) \subseteq U_j \subseteq U.$$

This proves that U is open. If $I = \emptyset$, the union is $\emptyset$, already covered by part (a).

- (c) **Finite intersections of open sets.** Let $U_1, \dots, U_k$ be open, where $k \geq 1$, and let $a \in \bigcap_{i=1}^k U_i$. For each i , choose $r_i > 0$ such that $B_\rho(a, r_i) \subseteq U_i$. Since there are only finitely many radii,

$$r = \min\{r_1, \dots, r_k\} > 0.$$

Then $B_\rho(a, r) \subseteq B_\rho(a, r_i) \subseteq U_i$ for every i , hence

$$B_\rho(a, r) \subseteq \bigcap_{i=1}^k U_i.$$

Thus the intersection is open. The intersection of the empty family of subsets of M is understood to be M , which is also open.

- (d) **Closed sets.** Let $(F_i)_{i \in I}$ be closed. By De Morgan's law,

$$M \setminus \bigcap_{i \in I} F_i = \bigcup_{i \in I} (M \setminus F_i).$$

Each set on the right is open. Part (b) shows that their union is open, so $\bigcap_{i \in I} F_i$ is closed.

For finitely many closed sets $F_1, \dots, F_k$,

$$M \setminus \bigcup_{i=1}^k F_i = \bigcap_{i=1}^k (M \setminus F_i).$$

The right-hand side is open by part (c), so the finite union is closed. The empty-family cases follow from part (a).

Why the finiteness assumptions matter

In $\mathbb{R}$ with the usual metric,

$$\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$$

is not open, even though every interval in the intersection is open. Likewise,

$$\bigcup_{n=1}^{\infty} [1/n, 1] = (0, 1]$$

is not closed, even though every interval in the union is closed.

3. Balls, singletons, and isolated points

Parts (a)–(b) are Exercise 15 in Lecture 1; part (c) is Exercise 31 in Lecture 2.

- (a) **Open balls are open: true.** Fix $x \in B(a, r)$. Since $d(a, x) < r$, the number $\varepsilon = r - d(a, x)$ is positive. If $y \in B(x, \varepsilon)$, the triangle inequality gives

$$d(a, y) \leq d(a, x) + d(x, y) < d(a, x) + \varepsilon = r.$$

Hence $y \in B(a, r)$, and $B(x, \varepsilon) \subseteq B(a, r)$. Every point of $B(a, r)$ therefore has a ball contained in $B(a, r)$.

- (b) **Closed balls are closed: true.** We prove that the complement of $B'(a, r)$ is open. Take $x \notin B'(a, r)$, so $d(a, x) > r$, and choose $\varepsilon = d(a, x) - r > 0$. For $y \in B(x, \varepsilon)$, the triangle inequality implies

$$d(a, y) \geq d(a, x) - d(x, y) > d(a, x) - \varepsilon = r.$$

Thus $y \notin B'(a, r)$, and the entire ball $B(x, \varepsilon)$ lies in its complement. The complement is open, proving the claim.

- (c) **Singletons are closed: true.** Let $a \in M$. For every $x \in M \setminus \{a\}$, the number $r_x = d(x, a)$ is positive, and

$$B(x, r_x) \subseteq M \setminus \{a\},$$

because a is at distance exactly r_x from x , whereas the ball uses a strict inequality. The complement of $\{a\}$ is therefore open, so $\{a\}$ is closed.

- (d) **If every point of M is isolated, every subset is open.** As in Lecture 2, p. 3, an **isolated point** $a \in M$ has $B(a, r_a) = \{a\}$ for some $r_a > 0$. This holds for every $a \in M$ by assumption. Let $U \subseteq M$ be arbitrary. For each $a \in U$,

$$B(a, r_a) = \{a\} \subseteq U,$$

so U is open. Its complement is another subset of M , hence is also open. Consequently, every subset is both open and closed.

The resulting family of open sets is the **discrete topology** (Lecture 2, Definition 30 and the examples following it). This does not require the metric itself to take only the values 0 and 1. For example, $\mathbb{Z}$ with the usual distance has $B(n, 1) = \{n\}$ for every integer n .

A closed ball need not be the closure of the open ball

The *closure* of a set is the smallest closed set containing it. The notation $B'(a, r)$ here means $\{x : d(a, x) \leq r\}$. In a discrete metric space with at least two points, $B(a, 1) = \{a\}$ is already closed, while $B'(a, 1) = M$. Thus the two notions can differ in a general metric space.

Terminology and results: continuous maps

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $F : X \rightarrow Y$ is **continuous at** $a \in X$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X : d_X(x, a) < \delta \implies d_Y(F(x), F(a)) < \varepsilon.$$

It is **continuous** if it is continuous at every point of X . Equivalently, for each $\varepsilon > 0$ some $\delta > 0$ satisfies $F(B_X(a, \delta)) \subseteq B_Y(F(a), \varepsilon)$, as in Lecture 1, pp. 8–9. If $X \subseteq \mathbb{R}^n$, we use the **induced subspace**: the Euclidean metric restricted to $X \times X$ (Lecture 1, p. 5 and Exercise 22). Only points $x \in X$ are considered, even at boundary points.

(a) **Composition.** If $F : X \rightarrow Y$ is continuous at a and $G : Y \rightarrow Z$ is continuous at $F(a)$, then $G \circ F$ is continuous at a . Indeed, given $\varepsilon > 0$, continuity of G gives $\eta > 0$ such that $d_Z(y, F(a)) < \eta$ implies $d_Z(G(y), G(F(a))) < \varepsilon$. Continuity of F then gives $\delta > 0$ such that $d_X(x, a) < \delta$ implies $d_Y(F(x), F(a)) < \eta$. Combining the implications proves the claim.

(b) **Preimages of open and closed sets.** By Proposition 23 in Lecture 1, F is continuous if and only if $F^{-1}(V)$ is open in X for every open set $V \subseteq Y$.

To prove the forward direction, take $a \in F^{-1}(V)$. Since $F(a) \in V$ and V is open, choose $\varepsilon > 0$ with $B_Y(F(a), \varepsilon) \subseteq V$. Continuity gives $\delta > 0$ with $F(B_X(a, \delta)) \subseteq B_Y(F(a), \varepsilon)$. Hence $B_X(a, \delta) \subseteq F^{-1}(V)$.

Conversely, suppose all open preimages are open. For any $a \in X$ and $\varepsilon > 0$, the set $F^{-1}(B_Y(F(a), \varepsilon))$ is open and contains a . It therefore contains $B_X(a, \delta)$ for some $\delta > 0$, which is exactly the continuity condition at a .

Equivalently, F is continuous if and only if the preimage of every closed subset of Y is closed in X (Lecture 1, Exercise 24). This follows by taking complements and using $F^{-1}(Y \setminus C) = X \setminus F^{-1}(C)$.

(c) **Heine's characterization of continuity.** By Theorem 27 in Lecture 1, F is continuous at a if and only if every sequence (x_n) in X converging to a satisfies $F(x_n) \rightarrow F(a)$.

If F is continuous, choose δ for a given $\varepsilon > 0$. Eventually $d_X(x_n, a) < \delta$, so eventually $d_Y(F(x_n), F(a)) < \varepsilon$.

For the converse, if F is not continuous at a , there exists $\varepsilon_0 > 0$ such that for each $n \geq 1$ one can choose $x_n \in X$ satisfying

$$d_X(x_n, a) < 1/n, \quad d_Y(F(x_n), F(a)) \geq \varepsilon_0.$$

Then $x_n \rightarrow a$, but $F(x_n)$ does not converge to $F(a)$, contradicting the stated sequence property.

4. Continuity of polynomials

First consider a coordinate projection $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $\pi_i(x) = x_i$. For the Euclidean metric,

$$|\pi_i(x) - \pi_i(a)| = |x_i - a_i| \leq \sqrt{\sum_{j=1}^n (x_j - a_j)^2} = d_2(x, a).$$

Thus each coordinate projection is continuous: for a given $\varepsilon > 0$, we may take $\delta = \varepsilon$. Constant functions are continuous as well.

We also use that finite sums and products of continuous real-valued functions are continuous. Here is a direct justification of the two-function case. If u, v are continuous at a , then

$$|(u(x) + v(x)) - (u(a) + v(a))| \leq |u(x) - u(a)| + |v(x) - v(a)|.$$

Choose a common $\delta > 0$ so that both terms on the right are less than $\varepsilon/2$ whenever $d(x, a) < \delta$. This proves continuity of the sum.

For products, continuity of u gives a neighbourhood of a on which $|u(x) - u(a)| < 1$, hence $|u(x)| \leq |u(a)| + 1 =: C$. On this neighbourhood,

$$\begin{aligned} |u(x)v(x) - u(a)v(a)| &\leq |u(x)| |v(x) - v(a)| + |v(a)| |u(x) - u(a)| \\ &\leq C |v(x) - v(a)| + |v(a)| |u(x) - u(a)|. \end{aligned}$$

Choose $\delta > 0$ small enough that this bound applies and that

$$|v(x) - v(a)| < \frac{\varepsilon}{2C}, \quad |u(x) - u(a)| < \frac{\varepsilon}{2(|v(a)| + 1)}.$$

Then the displayed bound is less than ε . This proves continuity of the product. Repeating these arguments gives the corresponding statements for any finite sum or product.

Now a polynomial in two variables has the form

$$P(x, y) = \sum_{(i,j) \in A} c_{ij} x^i y^j,$$

where A is a finite set of pairs of nonnegative integers and $c_{ij} \in \mathbb{R}$. The functions $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are coordinate projections. Their nonnegative integer powers are continuous by repeated multiplication (with the zeroth power the constant function 1). Every term $c_{ij} x^i y^j$ is therefore continuous, and so is their finite sum.

Conclusion in any number of variables

Every polynomial is a finite sum of monomials

$$c x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad c \in \mathbb{R}, \quad \alpha_i \in \{0, 1, 2, \dots\}.$$

Each monomial is a product of constants and coordinate functions. Consequently, every polynomial $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous everywhere.

5. Domains, continuity, level sets, and extrema

Openness and closedness of domains below refer to the ambient space $\mathbb{R}^2$ with its Euclidean metric. Continuity on a domain uses the induced metric. The **level set** at height $c \in \mathbb{R}$ is $L_c(F) = \{(x, y) \in D_F : F(x, y) = c\}$.

(a) **The function** $f(x, y) = \sqrt{\ln(x - y)}$

Domain. The logarithm requires $x - y > 0$, and the square root requires $\ln(x - y) \geq 0$. Since $\ln t \geq 0$ exactly when $t \geq 1$,

$$D_f = \{(x, y) \in \mathbb{R}^2 : x - y \geq 1\}.$$

This is the half-plane $y \leq x - 1$, including its boundary line. It is **closed**, because it is the preimage of $[1, \infty)$ under the continuous function $(x, y) \mapsto x - y$. It is **not open**: $(1, 0) \in D_f$, but for every $\varepsilon > 0$ the point $(1 - \varepsilon/2, 0)$ is within distance ε and does not belong to D_f .

Continuity. The map $(x, y) \mapsto x - y$ is continuous and takes values in $[1, \infty)$ on D_f . The functions $\ln : [1, \infty) \rightarrow [0, \infty)$ and $\sqrt{\cdot} : [0, \infty) \rightarrow [0, \infty)$ are continuous. Their composition shows that f is continuous at every point of D_f , including the boundary $x - y = 1$.

Level sets. Since $f \geq 0$, there are no level sets at negative heights. For $c \geq 0$,

$$f(x, y) = c \iff \ln(x - y) = c^2 \iff x - y = e^{c^2}.$$

Thus

$$L_c(f) = \begin{cases} \{(x, y) : y = x - e^{c^2}\}, & c \geq 0, \\ \emptyset, & c < 0. \end{cases}$$

These are parallel straight lines of slope 1; the zero level is the boundary of the domain. Squaring is reversible here because $c \geq 0$.

Global extrema and range. The global minimum is 0, attained at every point of $x - y = 1$. For each $t \geq 0$, the point $(e^{t^2}, 0)$ belongs to the domain and

$$f(e^{t^2}, 0) = t.$$

Consequently, the range is $[0, \infty)$, and there is no global maximum: the function is unbounded above.

(b) **The function** $g(x, y) = \sqrt{\frac{1}{x^2} - \frac{y^2}{x^2} - 1}$

Domain. The denominators require $x \neq 0$. For such x , we have $x^2 > 0$, so

$$\frac{1}{x^2} - \frac{y^2}{x^2} - 1 = \frac{1 - x^2 - y^2}{x^2} \geq 0 \iff x^2 + y^2 \leq 1.$$

Therefore

$$D_g = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \neq 0\}.$$

It is the closed unit disk with its entire vertical diameter removed, including the two endpoints $(0, \pm 1)$.

The domain is **not open**: $(1, 0) \in D_g$, but the points $(1 + \varepsilon/2, 0)$ lie outside D_g and can be arbitrarily close to $(1, 0)$. It is **not closed**: $(1/n, 0) \in D_g$ for every $n \geq 1$, while $(1/n, 0) \rightarrow (0, 0) \notin D_g$. Here we use Proposition 29 in Lecture 1: a set is closed exactly when it contains the limits of all its convergent sequences whose limits lie in the ambient space.

Continuity. The rational function $(1 - x^2 - y^2)/x^2$ is continuous on $\{x \neq 0\}$ and nonnegative on D_g . Composition with the square root shows that g is continuous everywhere on D_g , including points of the unit circle that remain in the domain.

Level sets. For $c < 0$, $L_c(g) = \emptyset$. For $c \geq 0$,

$$\begin{aligned} g(x, y) = c &\iff \frac{1 - x^2 - y^2}{x^2} = c^2 \quad \text{with } x \neq 0 \\ &\iff (1 + c^2)x^2 + y^2 = 1 \quad \text{with } x \neq 0. \end{aligned}$$

Every point satisfying the last conditions belongs to D_g , since $x^2 + y^2 = 1 - c^2x^2 \leq 1$. Thus

$$L_c(g) = \{(x, y) : (1 + c^2)x^2 + y^2 = 1, x \neq 0\} \quad (c \geq 0).$$

These are ellipses with horizontal semiaxis $1/\sqrt{1 + c^2}$ and vertical semiaxis 1, with the two points $(0, \pm 1)$ removed. For $c = 0$, the ellipse is the unit circle with those points removed.

Global extrema and range. The global minimum is 0, attained exactly when $x^2 + y^2 = 1$ and $x \neq 0$. For every $c \geq 0$,

$$g\left(\frac{1}{\sqrt{1 + c^2}}, 0\right) = c,$$

so the range is $[0, \infty)$. There is no global maximum. For example, $g(1/n, 0) = \sqrt{n^2 - 1} \rightarrow \infty$, even though the domain is bounded.

(c) **The function** $h(x, y) = \log(\sqrt{y+1} - x)$

Write $\log = \log_b$ for a fixed base $b > 1$. The usual choices $b = e$ and $b = 10$ give the same domain, continuity, and extrema conclusions.

Domain. The square root requires $y + 1 \geq 0$, while the logarithm requires $\sqrt{y+1} - x > 0$. Hence

$$D_h = \{(x, y) \in \mathbb{R}^2 : y \geq -1, x < \sqrt{y+1}\}.$$

If $x < 0$, the second condition is automatic once $y \geq -1$. If $x \geq 0$, both sides of $x < \sqrt{y+1}$ are nonnegative and we may square the inequality. Consequently,

$$D_h = \{(x, y) : x < 0, y \geq -1\} \cup \{(x, y) : x \geq 0, y > x^2 - 1\}.$$

In particular, squaring without first distinguishing the sign of x would incorrectly exclude part of the domain.

The domain is **not open**: $(-1, -1) \in D_h$, but points $(-1, -1 - \varepsilon/2)$ are outside and arbitrarily close. It is **not closed**: $(0, -1 + 1/n) \in D_h$, but these points converge to $(0, -1)$, where the logarithm's argument would be zero.

Continuity. The function $(x, y) \mapsto \sqrt{y+1} - x$ is continuous on the closed half-plane $y \geq -1$. It is strictly positive on D_h . Since $\log_b$ is continuous on $(0, \infty)$, the composition is continuous on all of D_h , including the points $(x, -1)$ with $x < 0$.

Level sets. For every $c \in \mathbb{R}$, put $q = b^c > 0$. Then

$$h(x, y) = c \iff \sqrt{y+1} - x = q \iff \sqrt{y+1} = x + q.$$

The right-hand side must be nonnegative. Squaring therefore yields

$$L_c(h) = \{(x, y) : y = (x + b^c)^2 - 1, x \geq -b^c\}.$$

Conversely, these conditions imply $\sqrt{y+1} = x + b^c$, so the logarithm's argument is exactly $b^c > 0$; no extra points have been introduced. Each level set is the right-hand half of an upward-opening parabola, including its vertex $(-b^c, -1)$. The other half is excluded by the condition $x + b^c \geq 0$.

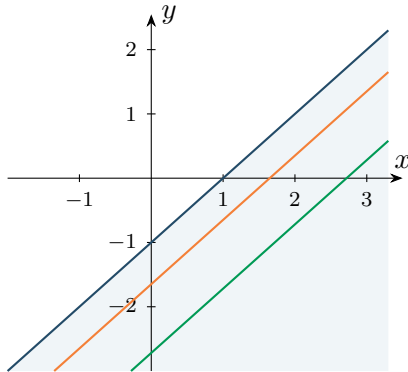
Global extrema and range. For every $c \in \mathbb{R}$, the point $(-b^c, -1)$ belongs to D_h and

$$h(-b^c, -1) = \log_b(b^c) = c.$$

Thus the range is all of $\mathbb{R}$, so h is unbounded both above and below. It has neither a global maximum nor a global minimum. For a base $0 < b < 1$, the same level-set formula and all these conclusions remain valid.

5. Geometric overview

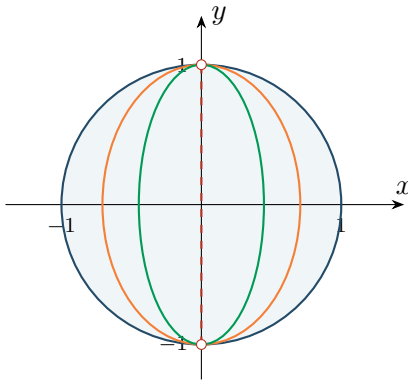
The shaded regions below are the domains. Solid boundary curves belong to the domain; dashed boundary curves and hollow points are excluded. The coloured curves show representative level sets.



f : a closed half-plane.

$$D_f = \{y \leq x - 1\}.$$

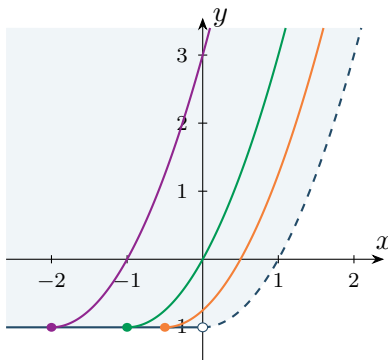
The level lines are $y = x - e^{c^2}$ for $c \geq 0$. Shown: $c = 0$, $c = 1/\sqrt{2}$, and $c = 1$. Larger values occur farther below the boundary line.



g : a disk with a diameter removed.

$$D_g = \{x^2 + y^2 \leq 1, x \neq 0\}.$$

The level ellipses satisfy $(1 + c^2)x^2 + y^2 = 1$ for $c \geq 0$. Shown: $c = 0$, $c = 1$, and $c = 2$. All omit $(0, \pm 1)$.



h : a partly included boundary.

$$D_h = \{y \geq -1, x < \sqrt{y+1}\}.$$

Each level is $y = (x + q)^2 - 1$, $x \geq -q$, where $q = b^c > 0$. Shown: $q = 1/2$, $q = 1$, and $q = 2$. Each vertex is included.