

Mathematical Analysis 2

Metric spaces 2

Peter Zeman

A. Open and closed sets

What is a neighbourhood? What are open and closed sets? What is a ball of radius r ?

1 Unit balls

What does the unit ball, that is, the set

$$B(a, 1) = \{x \in \mathbb{R}^n : d(a, x) < 1\},$$

look like in $\mathbb{R}^n$ with each of the following metrics?

- the Euclidean metric;
- the Manhattan (sum) metric;
- the maximum metric;
- the discrete metric.

Recall that on any nonempty set M , the discrete metric is defined by

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

2 Unions and intersections

Let (M, ρ) be a metric space. Justify the following statements.

- (a) M and $\emptyset$ are both open and closed in M .
- (b) The union of any family of open sets is open.
- (c) The intersection of finitely many open sets is open.
- (d) The analogous statements hold for closed sets: a finite union of closed sets is closed, and an arbitrary intersection of closed sets is closed.

3 Balls, singletons, and isolated points

Are the following statements true? Justify your answers.

- (a) For $r > 0$, the set $B(a, r) = \{x \in M : d(a, x) < r\}$ is open in the metric space (M, d) .
- (b) For $r > 0$, the set $B'(a, r) = \{x \in M : d(a, x) \leq r\}$ is closed in the metric space (M, d) .
- (c) Every singleton is closed in a metric space (M, d) .
- (d) Describe the open sets in a metric space (M, d) in which every point is isolated.

A point $a \in A \subseteq M$ is an *isolated point* of A if there exists $r > 0$ such that $B(a, r) \cap A = \{a\}$.

B. Continuity of real-valued functions of several variables

How is a continuous mapping defined? Which results about continuous mappings do you know? Consider composition, the characterization by preimages, and the characterization by convergent sequences.

4 Continuity of polynomials

Show that polynomials in several variables are continuous functions $P : \mathbb{R}^n \rightarrow \mathbb{R}$. For simplicity, you may restrict attention to polynomials in two variables.

5 Domains, continuity, level sets, and extrema

Investigate the following functions $f, g, h : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, where $\mathbb{R}^2$ is equipped, for example, with the Euclidean metric:

$$f(x, y) = \sqrt{\ln(x - y)},$$

$$g(x, y) = \sqrt{\frac{1}{x^2} - \frac{y^2}{x^2} - 1},$$

$$h(x, y) = \log(\sqrt{y + 1} - x).$$

For each function, determine its domain. Is the domain open or closed? Investigate continuity and describe the level sets. Does the function attain a global maximum or a global minimum on its domain?