Satisfiability of Commutative vs. Non-Commutative CSPs

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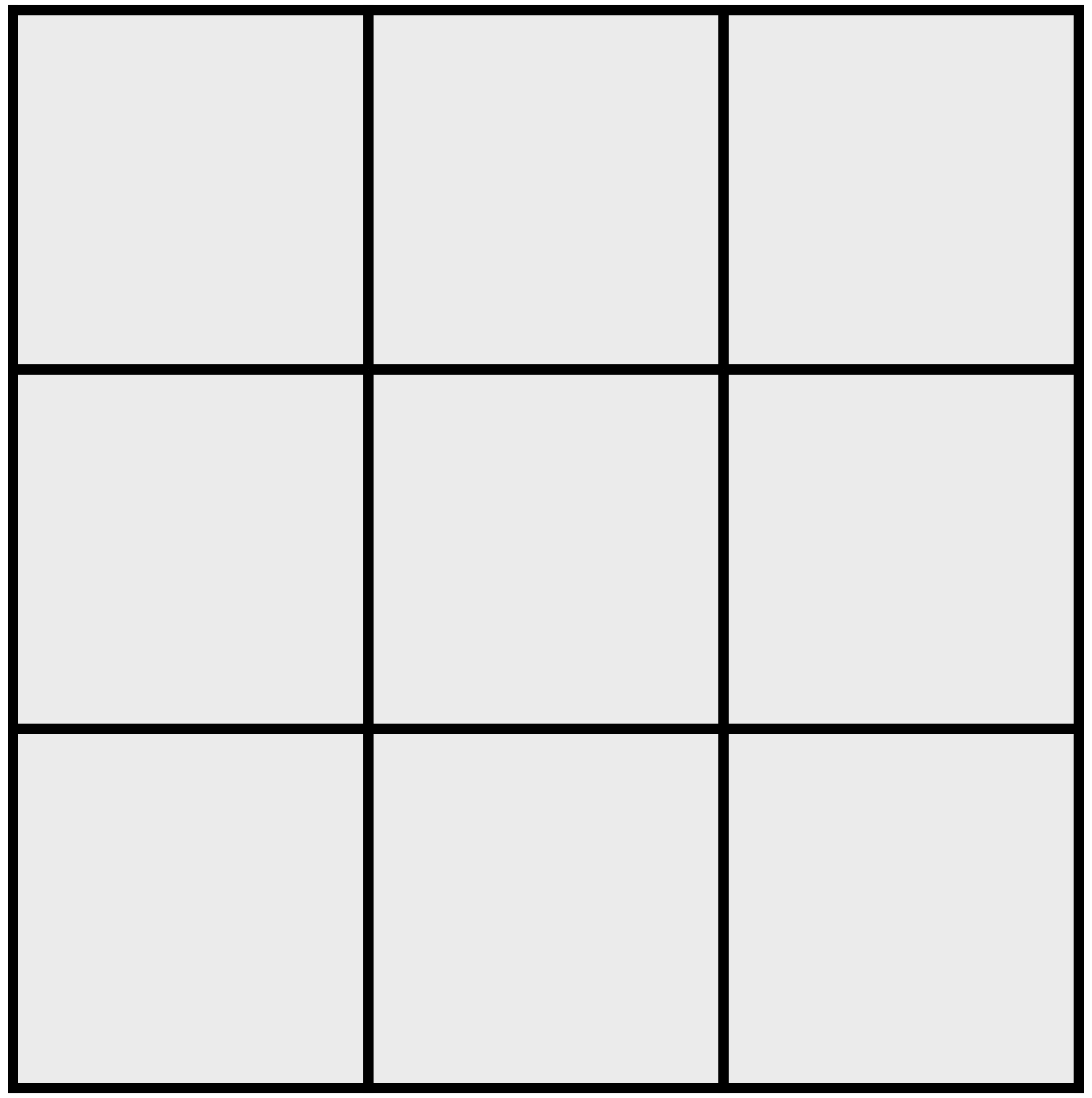
Magic Square Game



Magic Square Game

$+1$	$+1$	$+1$
$+1$	-1	-1
-1	$+1$	?

Magic Square Game

$+1$	$+1$	$+1$
$+1$	-1	-1
-1	$+1$	?

$$x_{11}x_{12}x_{13} = +1$$
$$x_{21}x_{22}x_{23} = +1$$
$$x_{31}x_{32}x_{33} = +1$$

$$x_{11}x_{21}x_{31} = -1$$
$$x_{12}x_{22}x_{32} = -1$$
$$x_{13}x_{23}x_{33} = -1$$

Magic Square Game

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$+1$	-1	-1
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$$+1 = -1$$

Magic Square Game

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$$x_{13}x_{23}x_{33} = -1$$

$$+1 = -1$$

$x_{ij} \in \mathbb{C}^{4x4}$

[Mermin-Peres'90]

Magic Square Game

+1	+1	+1
+1	−1	−1
−1	+1	?

$$x_{11}x_{12}x_{13} = +1$$

$$x_{21}x_{22}x_{23} = +1$$

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Bell-Kochen-Specker

$x_{ij} \in \mathbb{C}^{4x4}$

[Mermin-Peres'90]

CSPs

CSPs

2-SAT

$x_i \in \{0,1\}$ $\quad (x_1 \vee \overline{x}_3), (x_2 \vee x_4), (x_5 \vee \overline{x_4}), \ldots$

CSPs

2-SAT

$x_i \in \{0,1\}$ $\quad (x_1 \vee \overline{x_3}),\ (x_2 \vee x_4),\ (x_5 \vee \overline{x_4}),\ \ldots$

3-LIN

$x_i \in \{0,1\}$ $\quad (x_1 + x_2 = x_3),\ (x_2 + x_4 = x_5),\ (x_2 = 1),\ \ldots$

CSPs

$x_i \in \{0,1\} \qquad (x_1 \vee \overline{x}_3), (x_2 \vee x_4), (x_5 \vee \overline{x_4}), \ldots$ — 2-SAT

$x_i \in \{0,1\} \qquad (x_1 + x_2 = x_3), (x_2 + x_4 = x_5), (x_2 = 1), \ldots$ — 3-LIN

$v_i \in \{R, G, B\} \qquad (v_1 \neq v_3), (v_2 \neq v_4), (v_5 \neq v_4), \ldots$ — 3-COL

CSPs

$x_i \in \{0,1\}$ $(x_1 \vee \overline{x_3}), (x_2 \vee x_4), (x_5 \vee \overline{x_4}), \ldots$ 2-SAT

$x_i \in \{0,1\}$ $(x_1 + x_2 = x_3), (x_2 + x_4 = x_5), (x_2 = 1), \ldots$ 3-LIN

$v_i \in \{R, G, B\}$ $(v_1 \neq v_3), (v_2 \neq v_4), (v_5 \neq v_4), \ldots$ 3-COL

*(**V**ariables, **D**omain, **C**onstraints)*

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$x_i \in \{0,1\}$ $(x_1 \vee \overline{x_3}), (x_2 \vee x_4), (x_5 \vee \overline{x_4}), \ldots$ 2-SAT

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*(**V**ariables, **D**omain, **C**onstraints)*

$$R(x_1, \ldots, x_k)$$

$$R \subseteq D^k$$

CSPs

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*(**V**ariables, **D**omain, **C**onstraints)*

$$R(x_1, \ldots, x_k)$$

$$R \subseteq D^k$$

- Horn-SAT
- NAE-SAT
- 1-in-3-SAT
- Unreachability

CSPs

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*(**V**ariables, **D**omain, **C**onstraints)*

$$R(x_1, \ldots, x_k)$$

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CSP(Γ)

- Horn-SAT
- NAE-SAT
- 1-in-3-SAT
- Unreachability

Boolean CSPs

$\{-1, +1\}$

Boolean CSPs

$\{-1, +1\}$

Boolean CSPs

R

$\{(+1, +1)\}$

Boolean CSPs

$\{-1, +1\}$

R

$\{(+1, +1)\}$ →

x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

Boolean CSPs

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x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

→ p_R

$$\frac{1}{2}(x + y + xy - 1)$$

Boolean CSPs

$\{-1,+1\}$

R

$\{(+1,+1)\}$ →

x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

→ p_R

$$\frac{1}{2}(x+y+xy-1)$$

Which relations exhibit a gap between satisfiability and *operator* satisfiability?

Boolean CSPs

$\{-1, +1\}$

R

$\{(+1, +1)\}$ →

x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
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$$\frac{1}{2}(x + y + xy - 1)$$

$p_R(+1, -1, \ldots, -1) = +1$

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$\{(+1, +1)\}$ →

x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

→ p_R

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$p_R(A_1, A_2, \ldots, A_k) = I$

Which relations exhibit a gap between satisfiability and *operator* satisfiability?

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R		χ_R		p_R
$\{(+1, +1)\}$	→	(table below)	→	$\frac{1}{2}(x + y + xy - 1)$

x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

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R: $\{(+1, +1)\}$ → χ_R → p_R: $\frac{1}{2}(x + y + xy - 1)$

x	y	χ_R
-1	-1	-1
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

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Which relations exhibit a gap between satisfiability and *operator* satisfiability?

- 3-LIN ✓ *[Mermin-Peres'90]*

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-1	$+1$	-1
$+1$	-1	-1
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→ p_R

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Which relations exhibit a gap between satisfiability and *operator* satisfiability?

- 3-LIN ✔ *[Mermin-Peres'90]*
- 2-SAT ✘ *[Ji'13]*

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R → χ_R → p_R

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- 3-LIN ✓ *[Mermin-Peres'90]*
- 2-SAT ✗ *[Ji'13]*
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- Tractable CSPs except for 3-LIN ✗ *[Atserias-Kolaitis-Severini'19]*

Gaps for CSPs

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1. Unsatisfiable over *assignments* but satisfiable over *finite-dim operators*.

Gaps for CSPs

3-LIN
[Mermin-Peres'90]

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Gaps for CSPs

3-LIN
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Gaps for CSPs

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1 $\Rightarrow$ 2

Gaps for CSPs

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$1 \Rightarrow 2$

Gaps for CSPs

3-LIN
[Mermin-Peres'90]

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$$1 \Rightarrow 2 \Leftarrow 3$$

Gaps for CSPs

3-LIN
[Mermin-Peres'90]

3-LIN
[Slofstra'20]

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Gaps for CSPs

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Theorem *[Atserias-Kolaitis-Severini'19]*

Gaps for CSPs

3-LIN *[Mermin-Peres'90]*

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Theorem *[Atserias-Kolaitis-Severini'19]*

Let Γ be a set of relations on {-1,+1}. TFAE:

1. CSP(Γ) has no gap of the *first* kind.
2. CSP(Γ) has no gap of the *second* kind.
3. CSP(Γ) has no gap of the third kid.
4. Γ is -1-valid, +1-valid, 2-SAT, Horn-SAT, or Dual-Horn-SAT.

Gaps for CSPs

3-LIN *[Mermin-Peres'90]*

3-LIN *[Slofstra'20]*

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1. structure of 2-SAT, Horn-SAT
2. pp-reductions preserve gaps

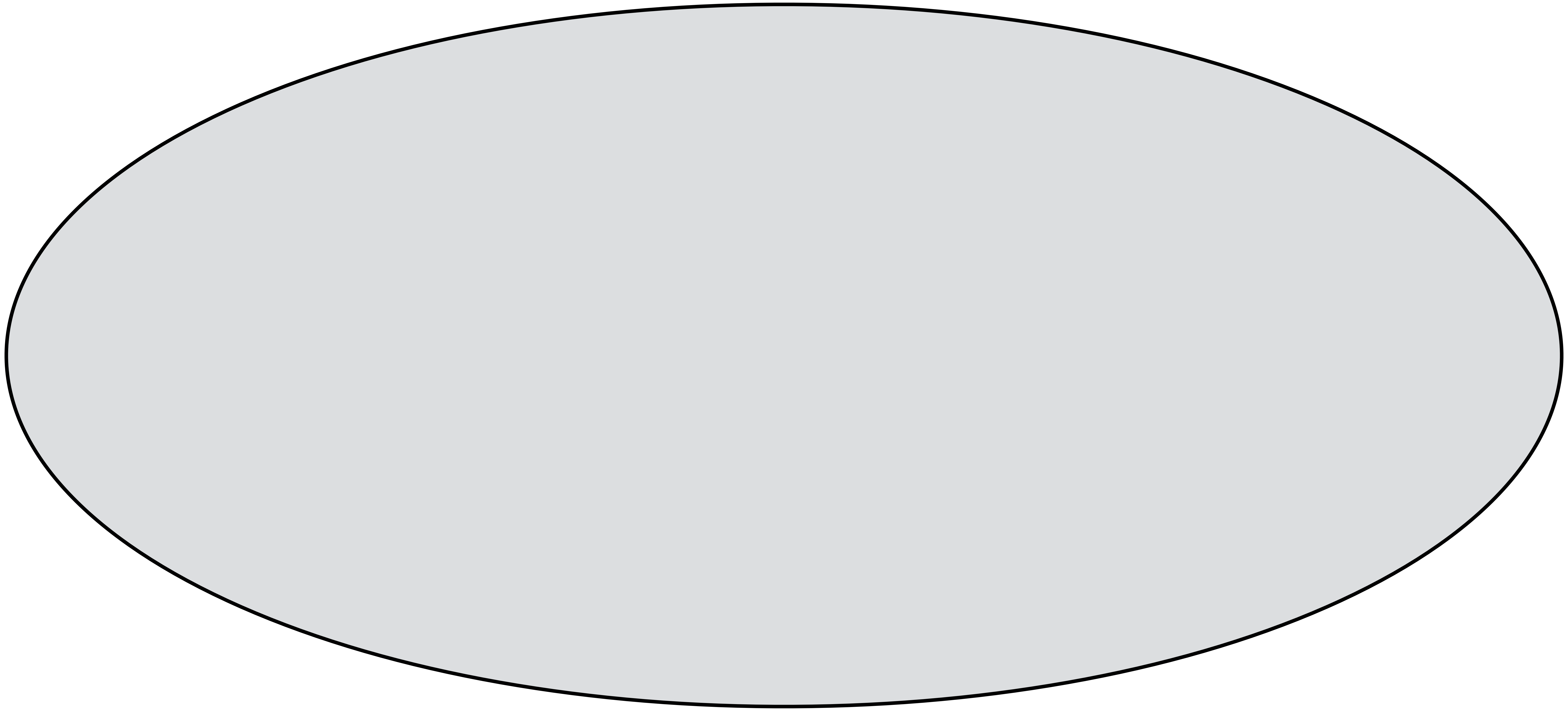
CSP(Γ)

CSP(Γ)

relations on U_d

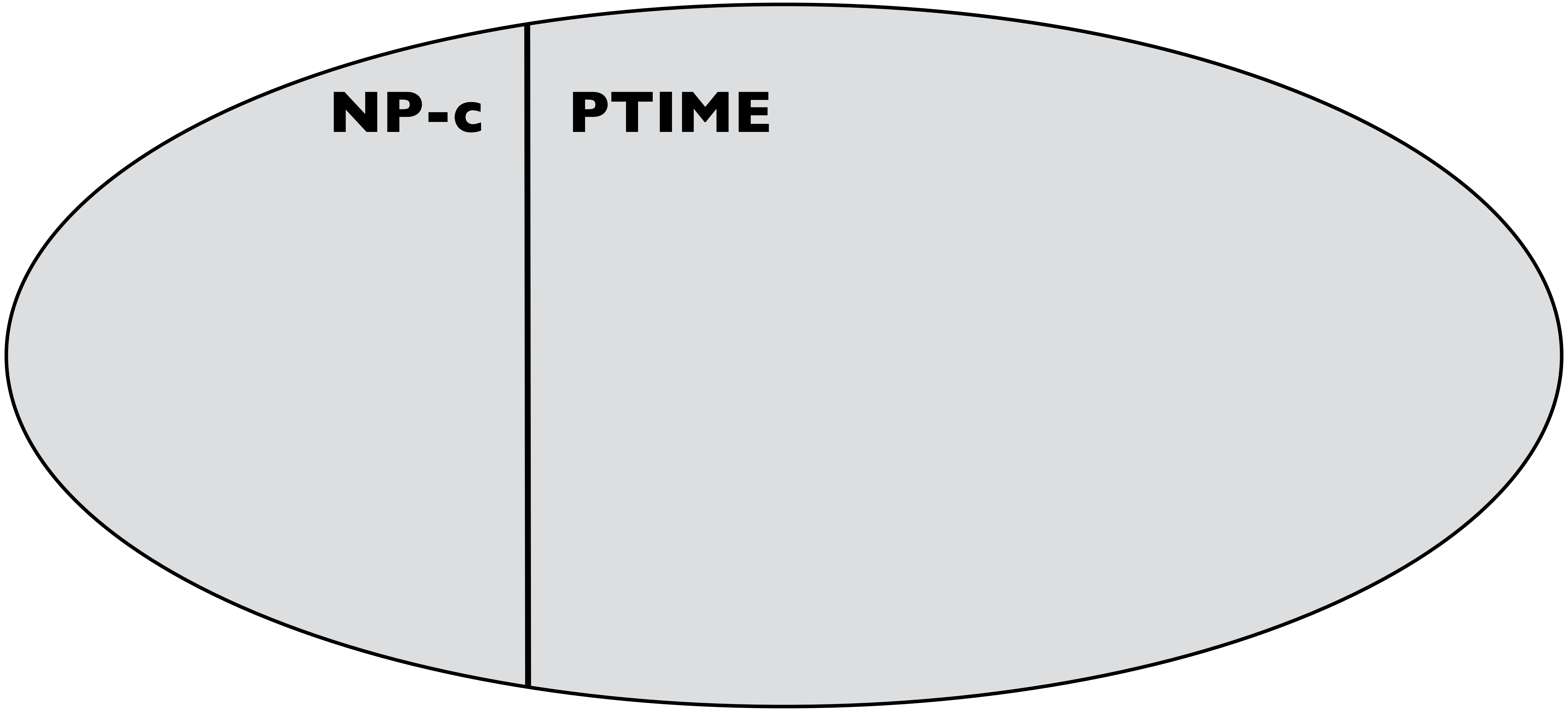
CSP(Γ)

relations on U_d



CSP(Γ)

relations on U_d



CSP(Γ)

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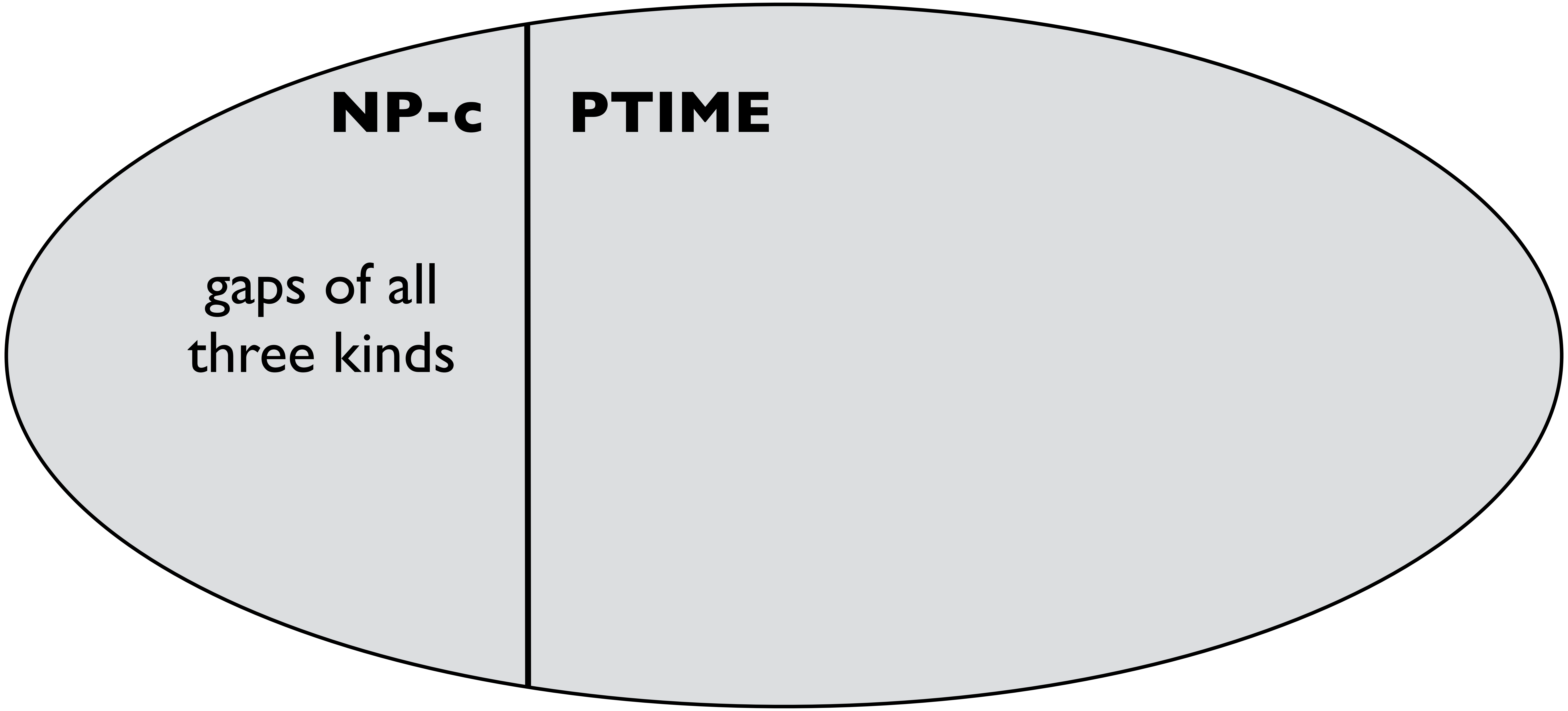

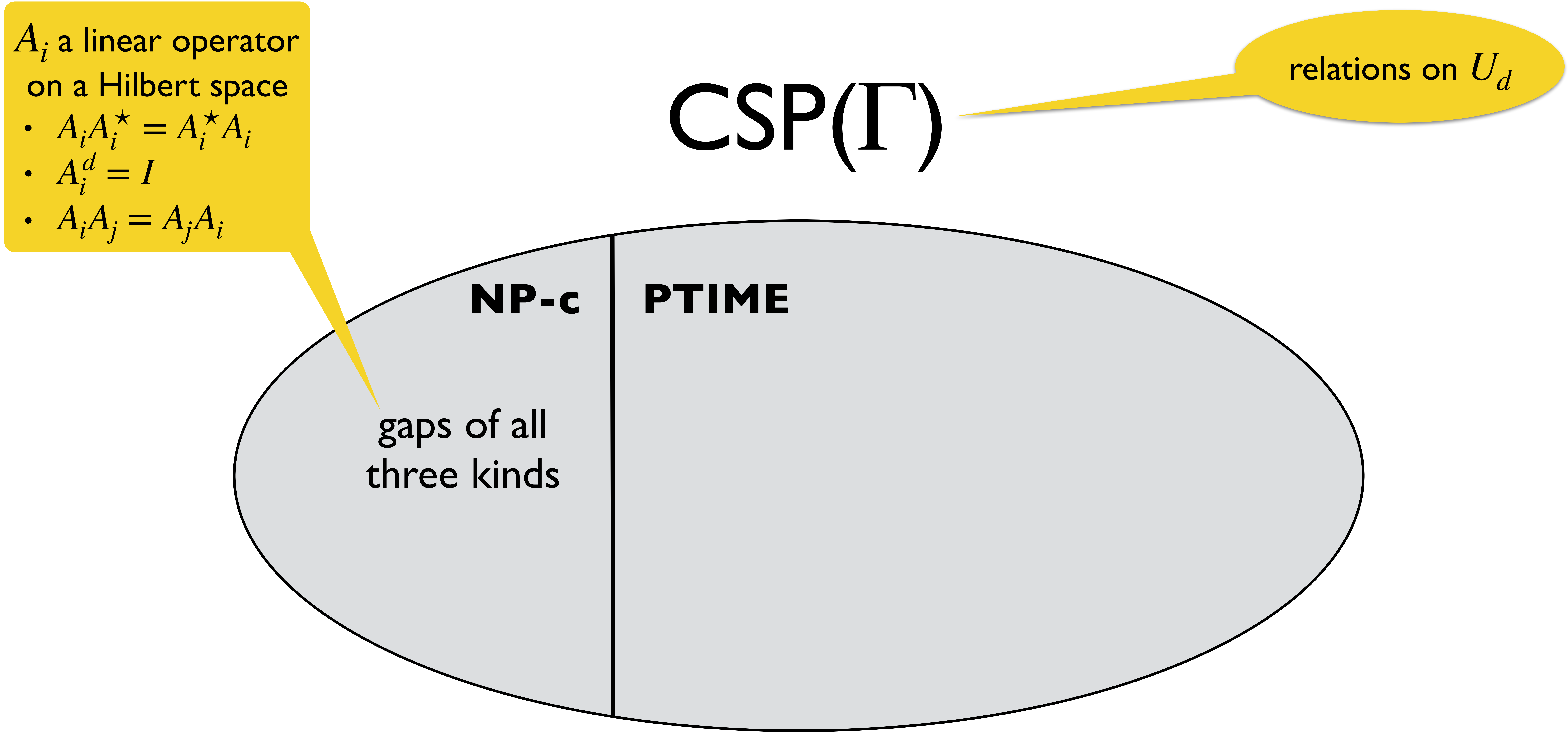
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• $A_i A_j = A_j A_i$
CSP(Γ)
relations on U_d
NP-c
PTIME
gaps of all three kinds

CSP(Γ)

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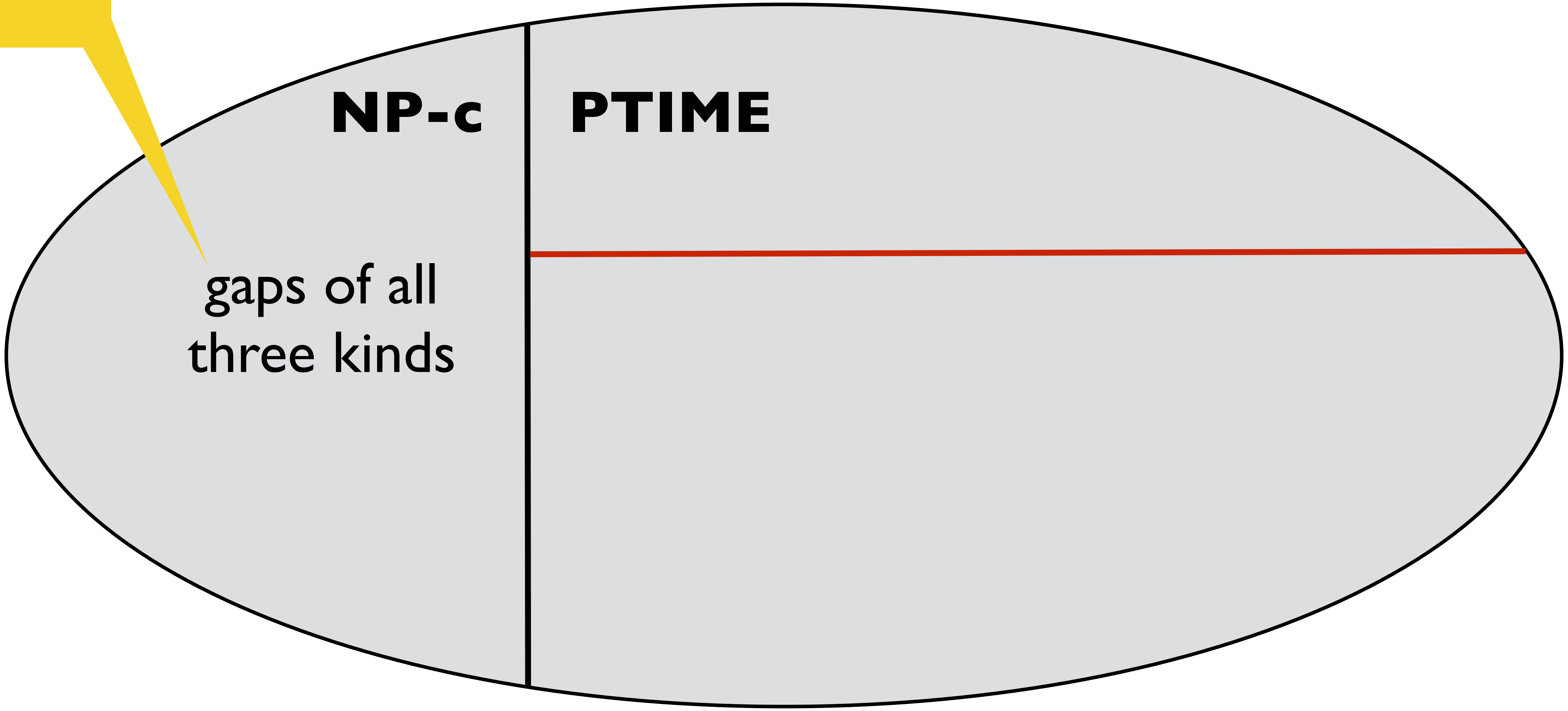

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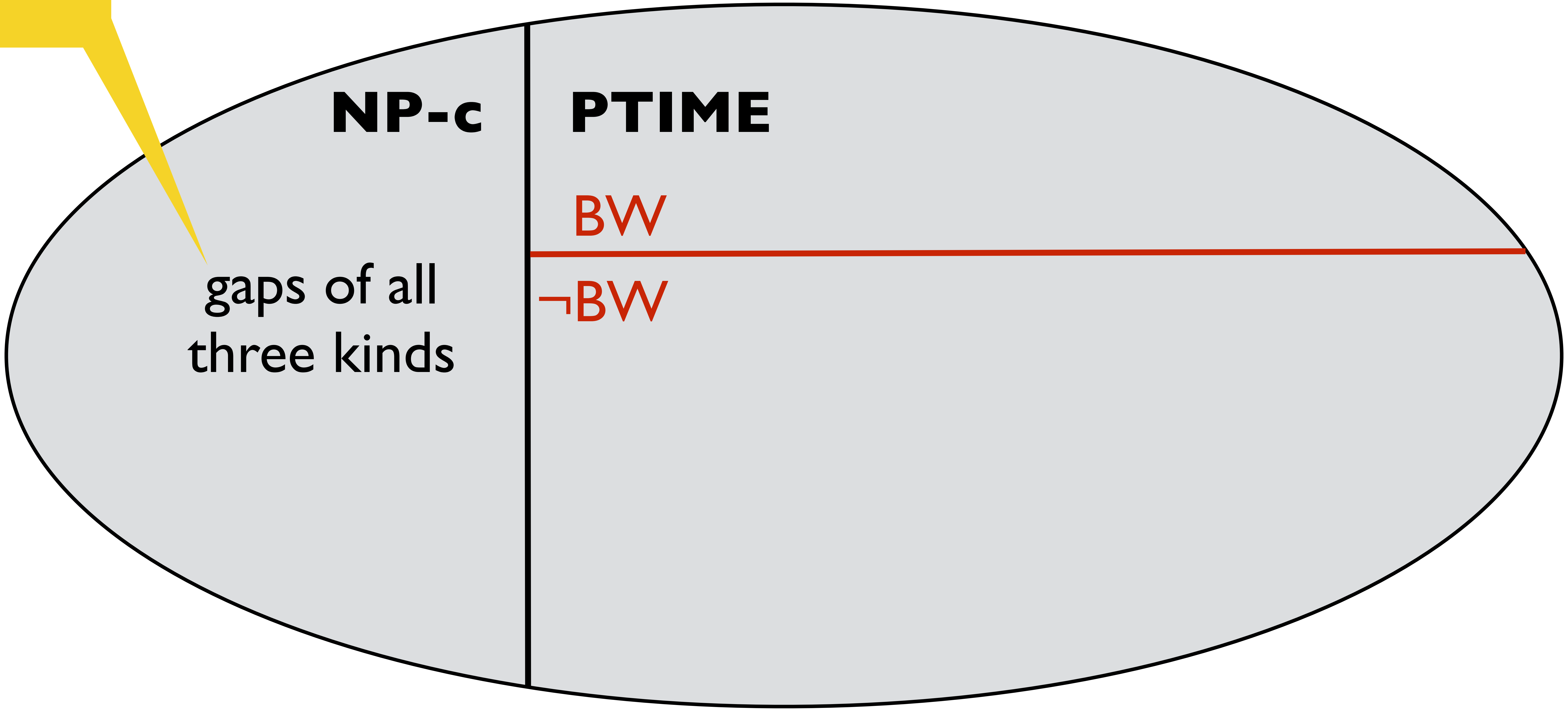
relations on U_d



CSP(Γ)

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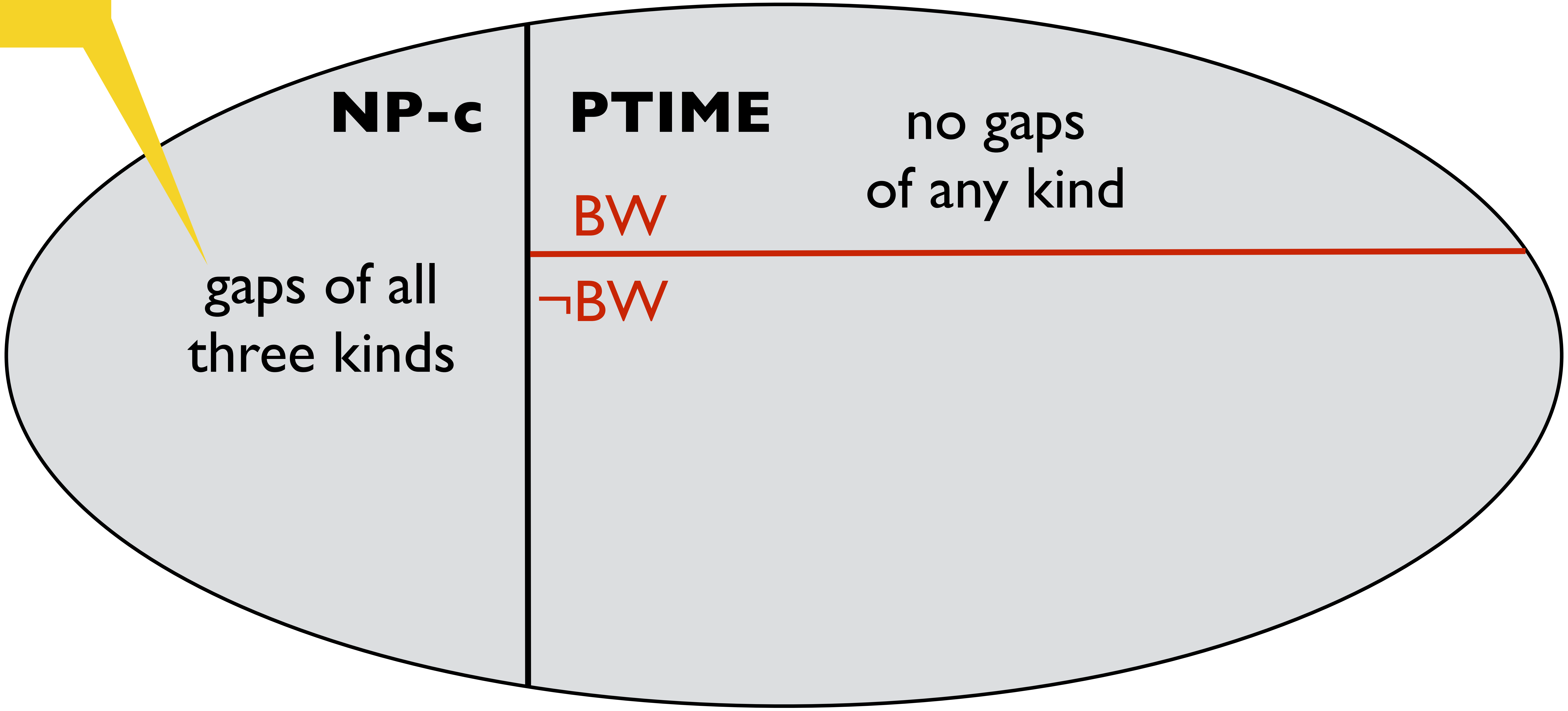

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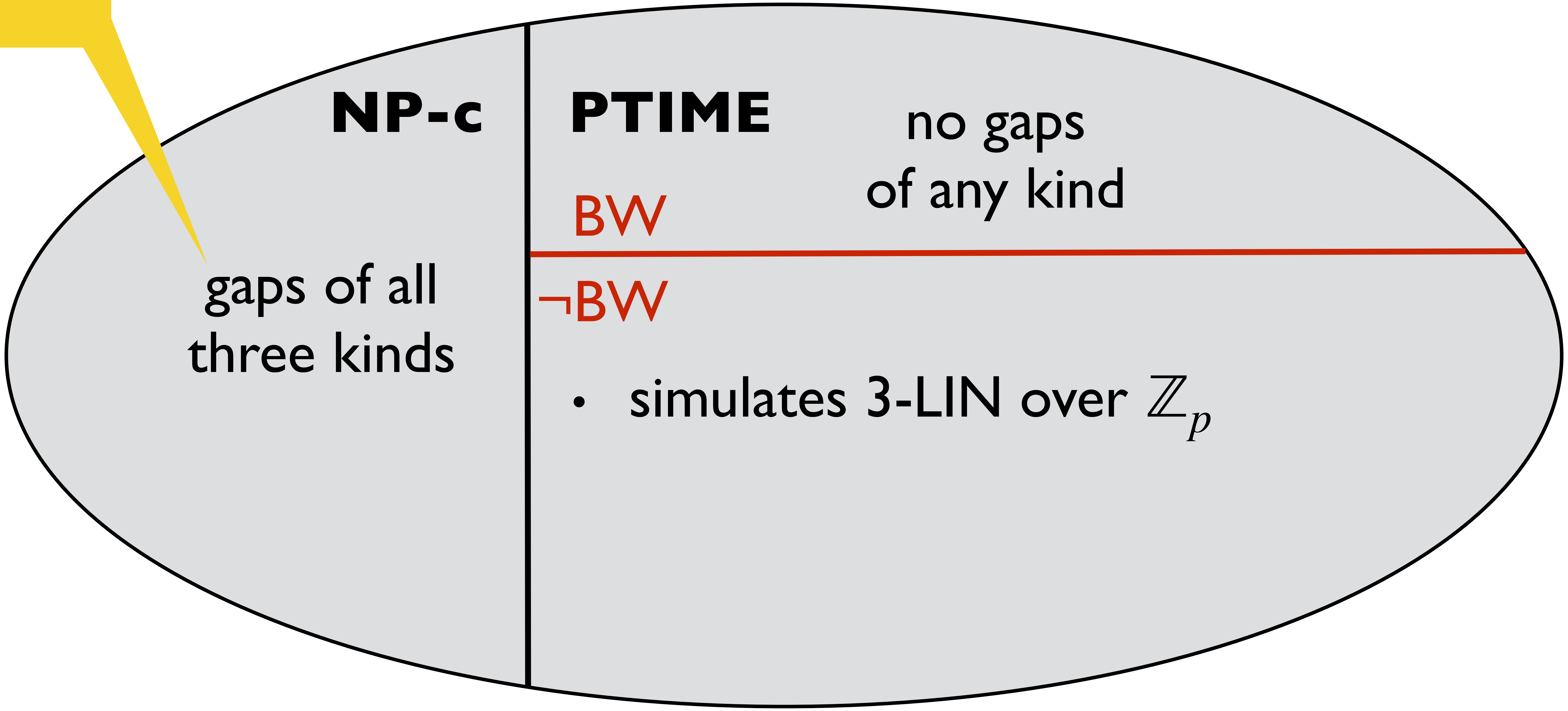

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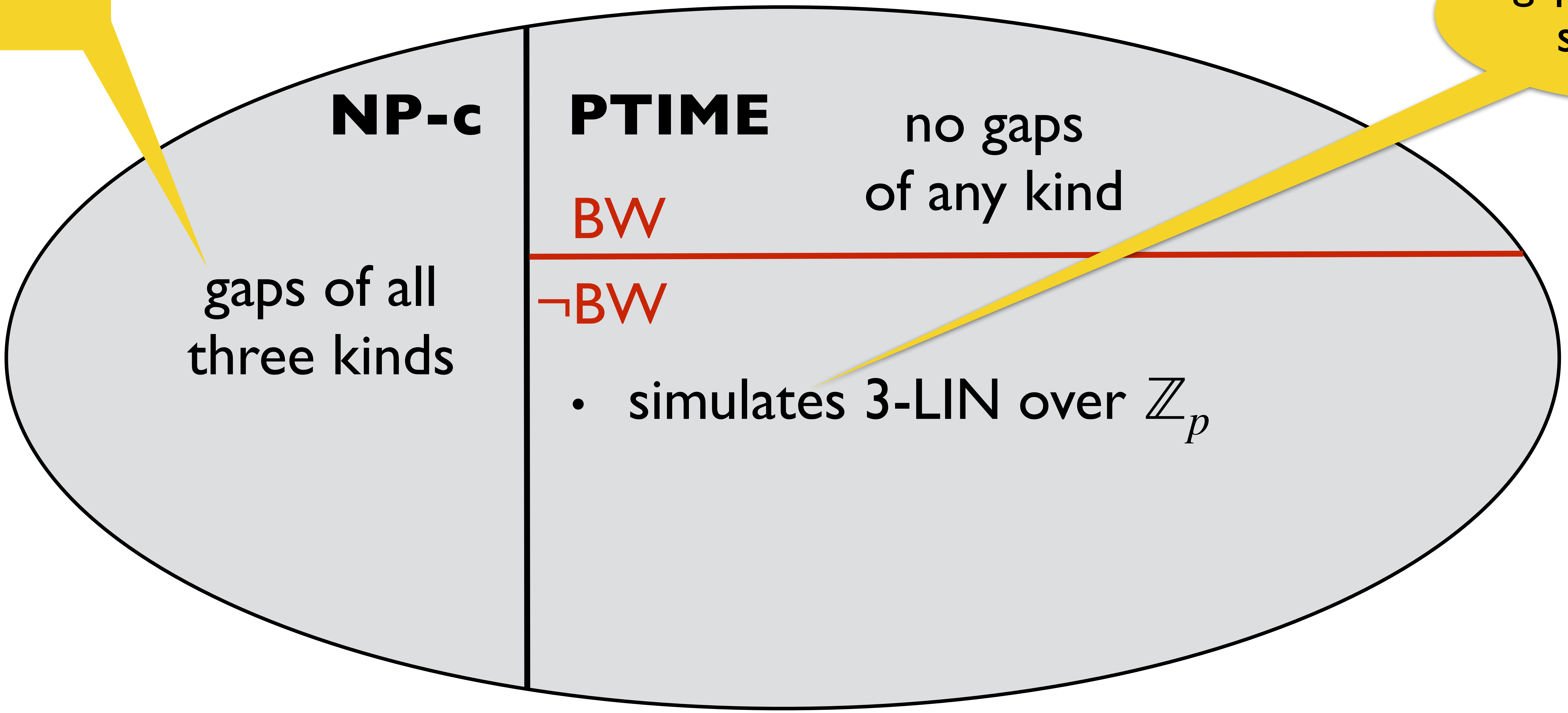
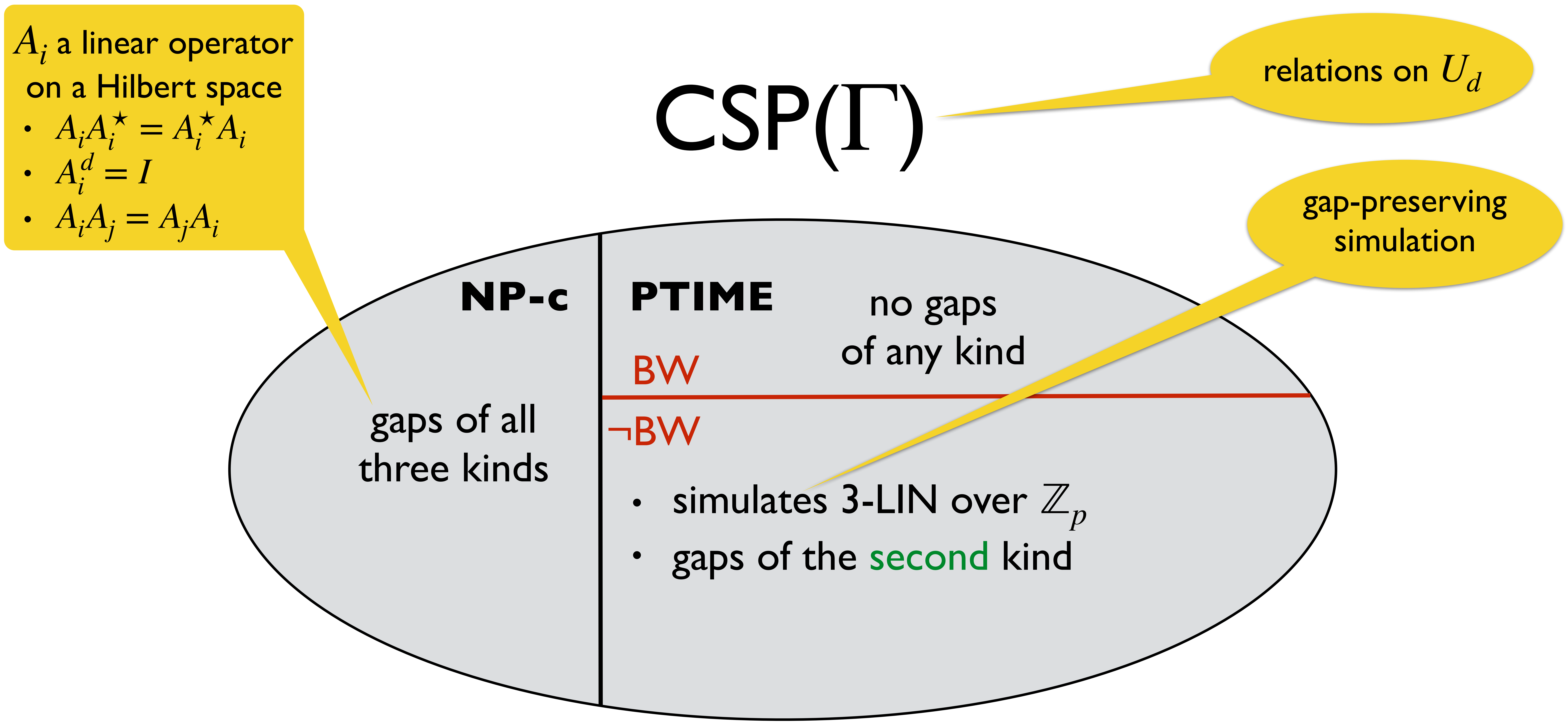
relations on U_d

gap-preserving simulation





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CSP(Γ)
relations on U_d
gap-preserving simulation
NP-c
PTIME
no gaps of any kind
BW
¬BW
gaps of all three kinds
• simulates 3-LIN over $\mathbb{Z}_p$
• gaps of the second kind

CSP(Γ)

A_i a linear operator on a Hilbert space
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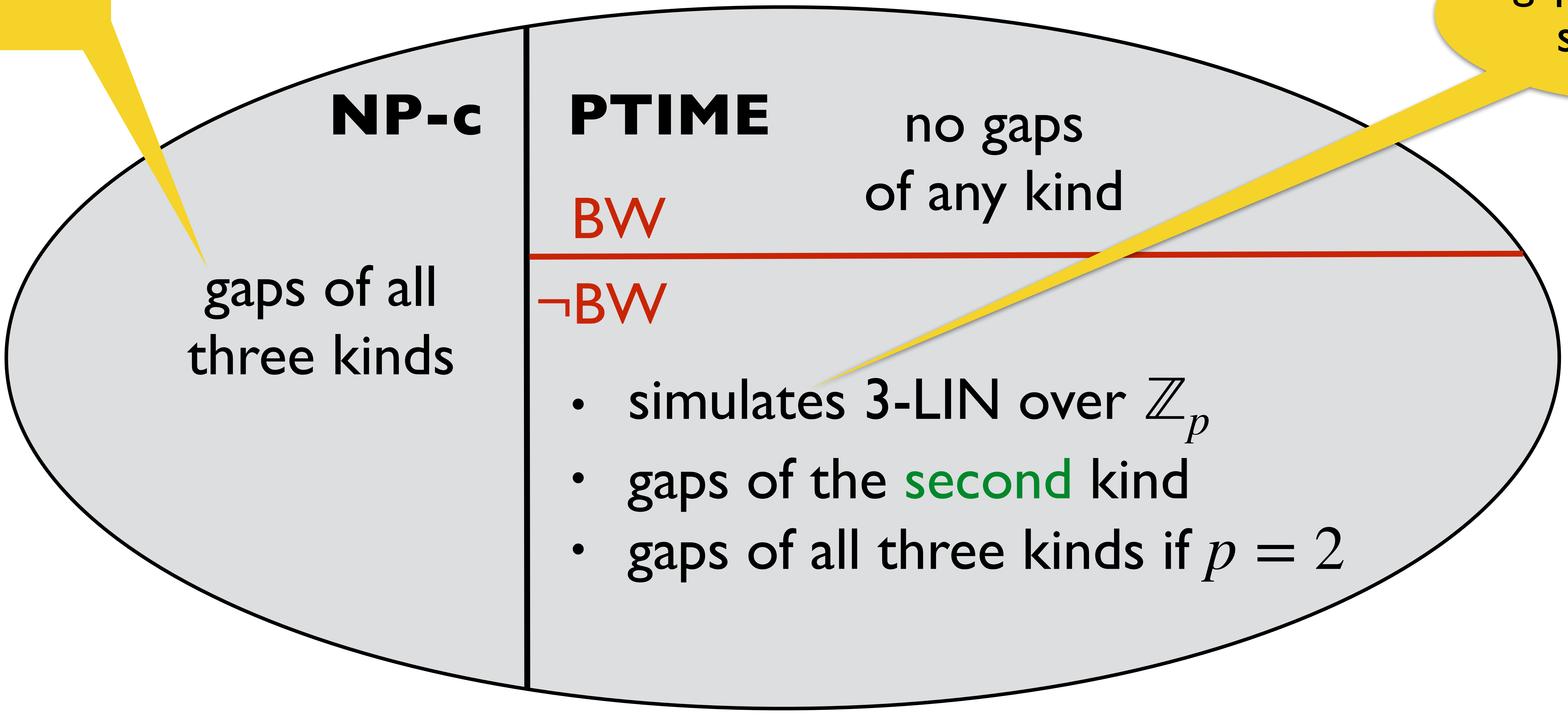

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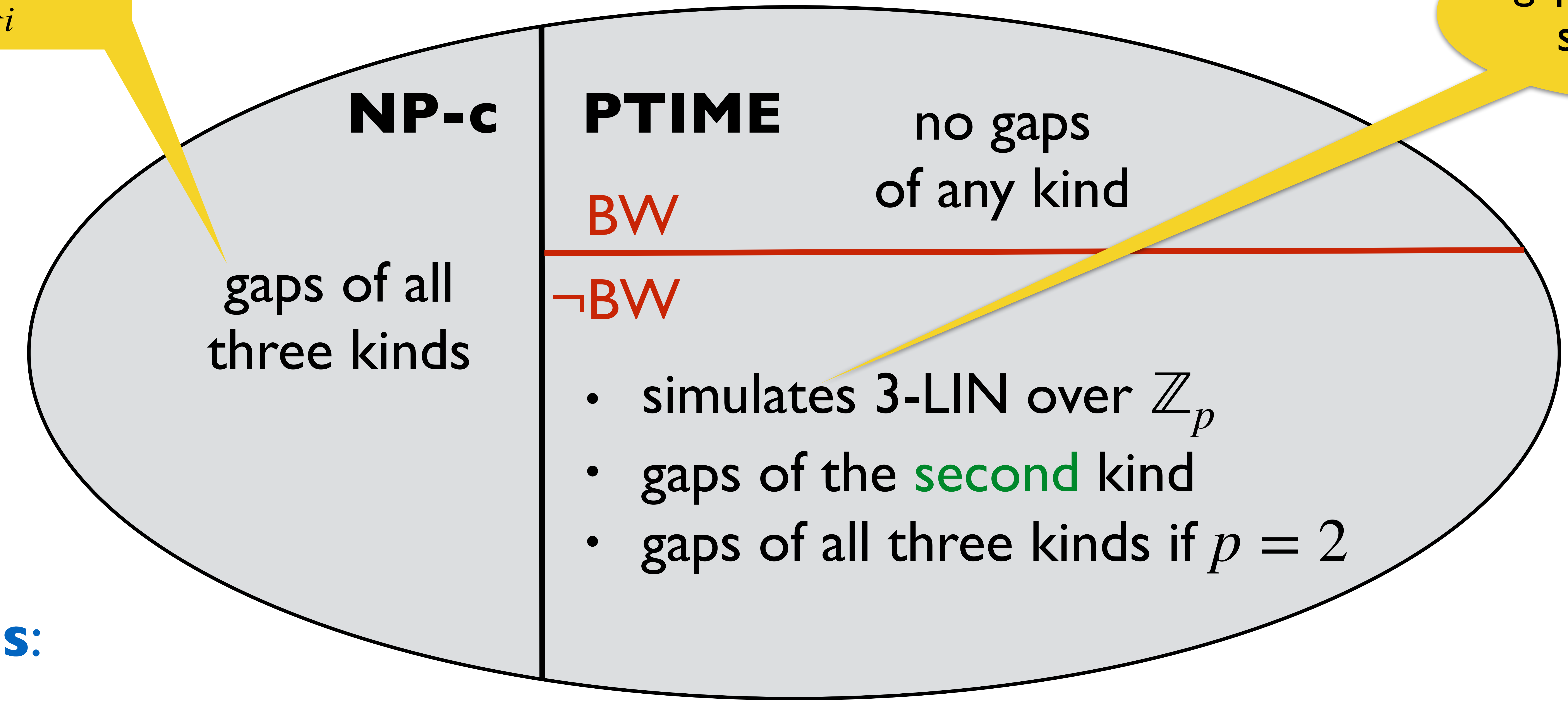
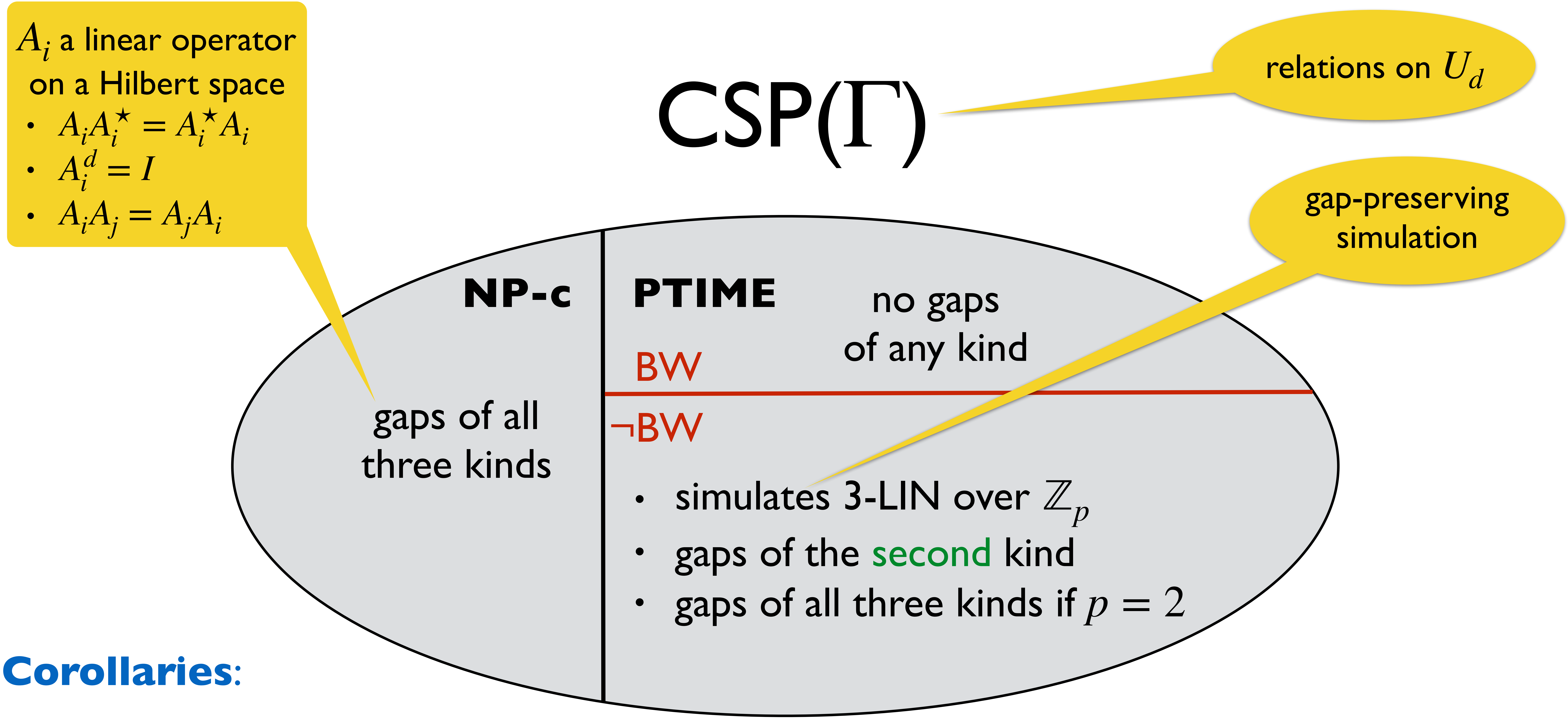


Corollaries:

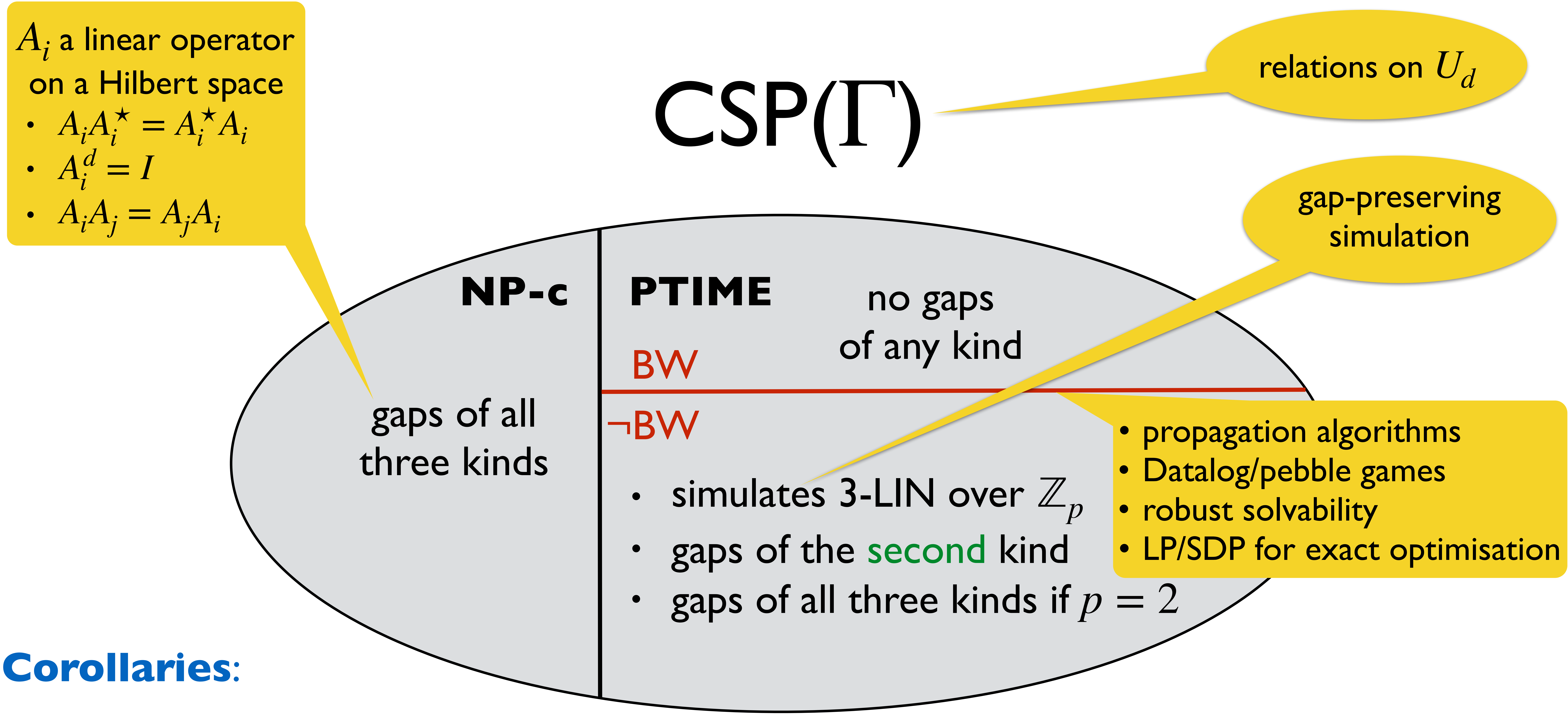
- $d = 2$ recovers AKS'19



Corollaries:

- $d = 2$ recovers AKS'19
- H-Colouring: bipartite H no gaps of any kind, o/w all three kinds

CSP(Γ)



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gap-preserving simulation

NP-c

gaps of all three kinds

PTIME

BW

no gaps of any kind

- propagation algorithms
- Datalog/pebble games
- robust solvability
- LP/SDP for exact optimisation

¬BW

- simulates 3-LIN over $\mathbb{Z}_p$
- gaps of the second kind
- gaps of all three kinds if $p = 2$

Corollaries:
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Open:
Gap of the first kind for 3-LIN over $\mathbb{Z}_p$ $(p > 2)$?

CSP(Γ)

A_i a linear operator on a Hilbert space
- $A_i A_i^\star = A_i^\star A_i$
- $A_i^d = I$
- $A_i A_j = A_j A_i$

relations on U_d

gap-preserving simulation

NP-c

gaps of all three kinds

PTIME

BW

no gaps of any kind

¬BW

- simulates 3-LIN over $\mathbb{Z}_p$
- gaps of the second kind
- gaps of all three kinds if $p = 2$

- propagation algorithms
- Datalog/pebble games
- robust solvability
- LP/SDP for exact optimisation

Corollaries:

- $d = 2$ recovers AKS'19
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no-go results!

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Idea:

- simulate some algorithm that solves BW CSPs (via polynomial equations)

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SLAC *[Kozik'21]*

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SLAC *[Kozik'21]*

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- simulate some algorithm that solves BW CSPs (via polynomial equations)
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- assume an operator solution for I, use the SLAC derivation from above, derive more polynomials, obtain a contradiction

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- pp-definitions

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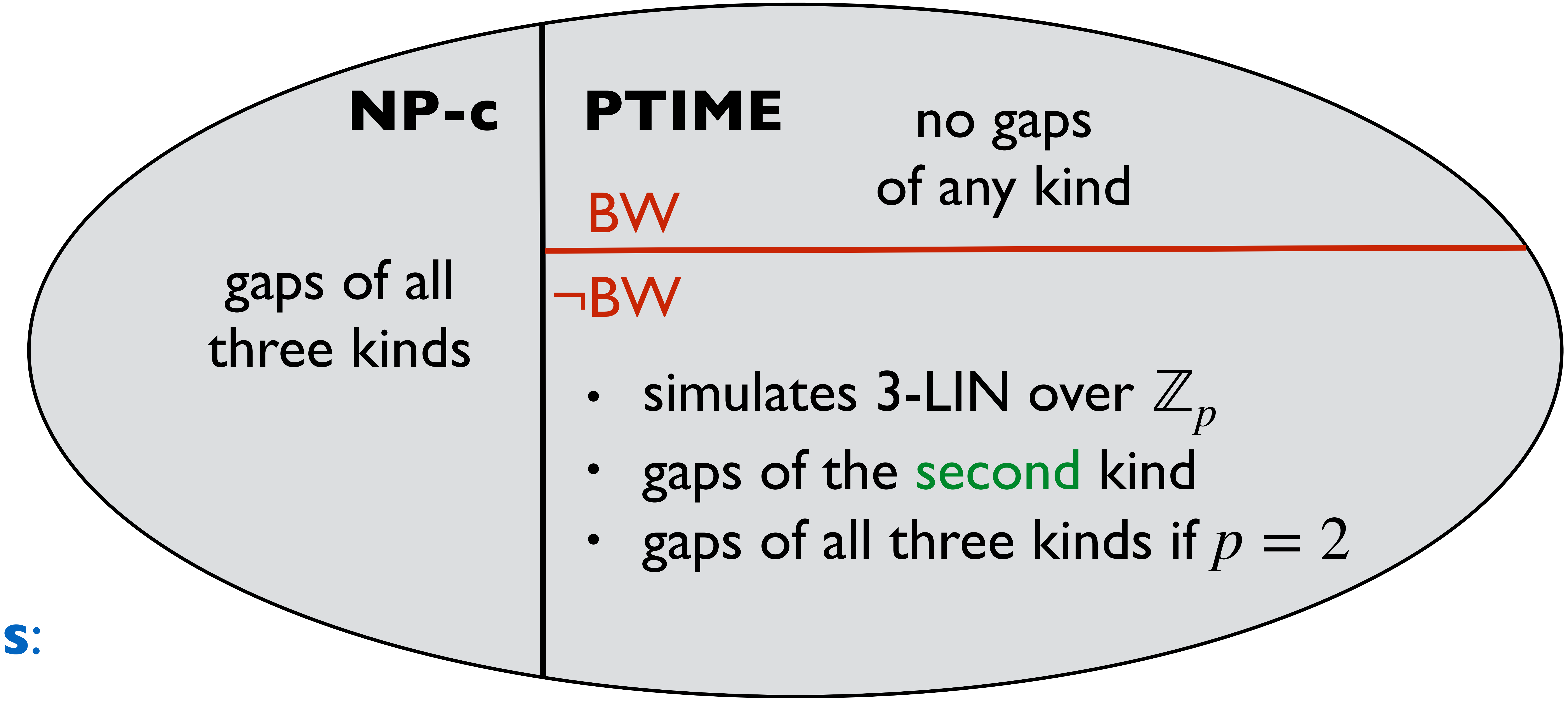
CSP(Γ)



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