NPA Hierarchy for Quantum Isomorphism and Homomorphism Indistinguishability

based on a joint work with D.E. Roberson, T. Seppelt, and P. Zeman.

Prem Nigam Kar

Institute of Mathematics of the Polish Academy of Sciences (IMPAN)

July 9, 2026

Venue: Quantum Constraint Satisfaction Problems, A Satellite workshop of ICALP 2026.

Introduction

Introduction

- A classic result of Lovasz [Lov67] says that two graphs are isomorphic if and only if they admit the same number of homomorphisms from any graph, which led to the notion of homomorphism indistinguishability.

Introduction

- A classic result of Lovasz [Lov67] says that two graphs are isomorphic if and only if they admit the same number of homomorphisms from any graph, which led to the notion of homomorphism indistinguishability.
- Two graphs G and H are said to be *homomorphism indistinguishable* over $\mathcal{F}$, written $G \cong_{\mathcal{F}} H$, if for all graphs $F \in \mathcal{F}$, $\hom(F, G) = \hom(F, H)$.

Introduction

- A classic result of Lovasz [Lov67] says that two graphs are isomorphic if and only if they admit the same number of homomorphisms from any graph, which led to the notion of homomorphism indistinguishability.
- Two graphs G and H are said to be *homomorphism indistinguishable* over $\mathcal{F}$, written $G \cong_{\mathcal{F}} H$, if for all graphs $F \in \mathcal{F}$, $\hom(F, G) = \hom(F, H)$.
- Homomorphism indistinguishability provides a unified framework for studying relaxations of graph isomorphism arising from a wide range of fields including model theory, optimization, algebraic graph theory, machine learning, and category theory.

Homomorphism Indistingushability

The class $\mathcal{F}$	**The relation** $G \cong_{\mathcal{F}} H$
All graphs	Isomorphism [Lovász 1967]
Cycles	Cospectrality
Cycles & paths	Cospectral & cospectral complements
Trees	Fractional isomorphism [Dvořák 2010]
Treewidth $\leq k$	Indistinguishable by k-WL [Dvořák 2010]
Treedepth $\leq k$	Ind. by FOL w/ counting of quantifier rank $\leq k$ [Grohe 2020]

Homomorphism Indistingushability

The class $\mathcal{F}$	The relation $G \cong_{\mathcal{F}} H$
All graphs	Isomorphism [Lovász 1967]
Cycles	Cospectrality
Cycles & paths	Cospectral & cospectral complements
Trees	Fractional isomorphism [Dvořák 2010]
Treewidth $\leq k$	Indistinguishable by k-WL [Dvořák 2010]
Treedepth $\leq k$	Ind. by FOL w/ counting of quantifier rank $\leq k$ [Grohe 2020]
Planar graphs	Quantum isomorphism [Mančinska & Roberson 2020]

Homomorphism Indistingushability

The class $\mathcal{F}$	The relation $G \cong_{\mathcal{F}} H$
All graphs	Isomorphism [Lovász 1967]
Cycles	Cospectrality
Cycles & paths	Cospectral & cospectral complements
Trees	Fractional isomorphism [Dvořák 2010]
Treewidth $\leq k$	Indistinguishable by k-WL [Dvořák 2010]
Treedepth $\leq k$	Ind. by FOL w/ counting of quantifier rank $\leq k$ [Grohe 2020]
Planar graphs	Quantum isomorphism [Mančinska & Roberson 2020]
$\mathcal{L}_k$	k^{th}-levek of Lasserre hierarchy is feasible. [Seppelt & Roberson 2024]

Homomorphism Indistingushability

The class $\mathcal{F}$	The relation $G \cong_{\mathcal{F}} H$
All graphs	Isomorphism [Lovász 1967]
Cycles	Cospectrality
Cycles & paths	Cospectral & cospectral complements
Trees	Fractional isomorphism [Dvořák 2010]
Treewidth $\leq k$	Indistinguishable by k-WL [Dvořák 2010]
Treedepth $\leq k$	Ind. by FOL w/ counting of quantifier rank $\leq k$ [Grohe 2020]
Planar graphs	Quantum isomorphism [Mančinska & Roberson 2020]
$\mathcal{L}_k$	k^{th}-levek of Lasserre hierarchy is feasible. [Seppelt & Roberson 2024]
$\mathcal{P}_k$	k^{th} level of NPA is feasible for (G, H)-isomorphism game

Main Result

Theorem [KRSZ26]

Let G, H be two graphs on the same number of vertices. Then, for each $k \in \mathbb{N}$, there is a class of planar graphs $\mathcal{P}_k$, such that the following are equivalent:

1. The k^{th}-level of the NPA hierarchy for the (G, H)-graph isomorphism game is feasible
2. G and H are homomorphism indistinguishable over $\mathcal{P}_k$, i.e., $G \cong_{\mathcal{P}_k} H$.

Consequences

Consequences

- As a corollary of our main result and the convergence of the NPA hierarchy, we are able to give an alternative proof of the homomorphism indistinguishability characterization for quantum isomorphism of graphs from [MR20].

Consequences

- As a corollary of our main result and the convergence of the NPA hierarchy, we are able to give an alternative proof of the homomorphism indistinguishability characterization for quantum isomorphism of graphs from [MR20].
- Another consequence of our main result is a randomized polynomial time algorithm deciding *exact feasibility* of each level of the NPA hierarchy for quantum isomorphism.

Consequences

- As a corollary of our main result and the convergence of the NPA hierarchy, we are able to give an alternative proof of the homomorphism indistinguishability characterization for quantum isomorphism of graphs from [MR20].
- Another consequence of our main result is a randomized polynomial time algorithm deciding *exact feasibility* of each level of the NPA hierarchy for quantum isomorphism.
- We are currently working on extending this algorithm to a larger class of semidefinite programs, inluding the NPA hierarchy of relaxations for Linear Constraint System games.

The Graph Isomorphism Game

The Graph Isomorphism Game

- The graph isomorphism game (introduced in [AMR$^+$19]) is a two-player cooperative game in which two spatially separated players Alice and Bob try to convince a referee/ verifier that two graphs are isomorphic.

The Graph Isomorphism Game

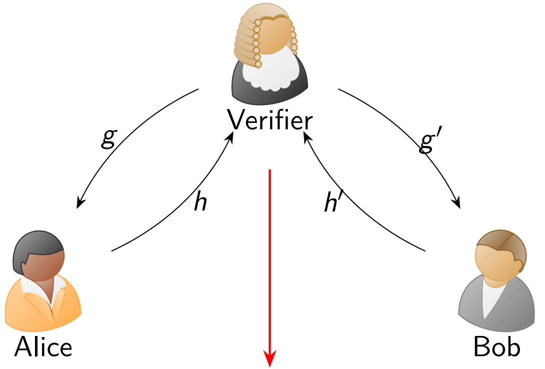
- The graph isomorphism game (introduced in [AMR$^+$19]) is a two-player cooperative game in which two spatially separated players Alice and Bob try to convince a referee/ verifier that two graphs are isomorphic.
- In each round, the referee picks two vertices g and g' from the graph G (uniformly at random) and sends them to Alice and Bob respectively. Alice and Bob are required to respond with vertices h and h' from the graph H respectively.

The Graph Isomorphism Game

- The graph isomorphism game (introduced in [AMR$^+$19]) is a two-player cooperative game in which two spatially separated players Alice and Bob try to convince a referee/ verifier that two graphs are isomorphic.
- In each round, the referee picks two vertices g and g' from the graph G (uniformly at random) and sends them to Alice and Bob respectively. Alice and Bob are required to respond with vertices h and h' from the graph H respectively.
- The referee then decides if they win or lose based on $V : V(G) \times V(G) \times V(H) \times V(H) \to \{0, 1\}$

$$V(h, h' \mid g, g') = \begin{cases} 1 & \text{if } \operatorname{rel}(g, g') = \operatorname{rel}(h, h') \\ 0 & \text{otherwise} \end{cases}$$

The Graph Isomorphism Game



$$V(h, h' \mid g, g')$$

Figure: The Graph Isomorphism Game

Quantum Strategies

Quantum Strategies

- A *quantum strategy* for the graph isomorphism game involves mutually commuting POVMs $\{E_{gh}\}_{h \in V(H)} \subseteq \mathcal{B}(\mathcal{H})$ and $\{F_{g'h'}\}_{h' \in V(H)} \subseteq \mathcal{B}(\mathcal{H})$ for each $g, g' \in V(G)$ and an entangled state $|\psi\rangle \in \mathcal{B}(\mathcal{H})$ shared between Alice and Bob.

Quantum Strategies

- A *quantum strategy* for the graph isomorphism game involves mutually commuting POVMs $\{E_{gh}\}_{h\in V(H)} \subseteq \mathcal{B}(\mathcal{H})$ and $\{F_{g'h'}\}_{h'\in V(H)} \subseteq \mathcal{B}(\mathcal{H})$ for each $g, g' \in V(G)$ and an entangled state $|\psi\rangle \in \mathcal{B}(\mathcal{H})$ shared between Alice and Bob.
- Upon receiving the questions g, g' measure their part of the shared entangled state $|\psi\rangle$ by using the POVMs $\{E_{g,h}\}_{h\in V(H)}$ and $\{F_{g'h'}\}_{h'\in V(H)}$ respectively. Hence, the probability that they output h, h' given the questions g, g', is given by

$$p(h, h' \mid g, g') = \langle \psi \mid E_{gh} F_{g'h'} \mid \psi \rangle .$$

Quantum Strategies

- A *quantum strategy* for the graph isomorphism game involves mutually commuting POVMs $\{E_{gh}\}_{h \in V(H)} \subseteq \mathcal{B}(\mathcal{H})$ and $\{F_{g'h'}\}_{h' \in V(H)} \subseteq \mathcal{B}(\mathcal{H})$ for each $g, g' \in V(G)$ and an entangled state $|\psi\rangle \in \mathcal{B}(\mathcal{H})$ shared between Alice and Bob.
- Upon receiving the questions g, g' measure their part of the shared entangled state $|\psi\rangle$ by using the POVMs $\{E_{g,h}\}_{h \in V(H)}$ and $\{F_{g'h'}\}_{h' \in V(H)}$ respectively. Hence, the probability that they output h, h' given the questions g, g', is given by

$$p(h, h' \mid g, g') = \langle \psi \mid E_{gh} F_{g'h'} \mid \psi \rangle .$$

- Such a quantum strategy is *perfect* if $V(h, h' \mid g, g') = 0$ implies that $p(h, h' \mid g, g') = 0$. In this case, the graphs are said to be *quantum isomorphic.*

The existence of perfect quantum strategies is characterized by the following result:

Quantum Isomorphism of Graphs

The existence of perfect quantum strategies is characterized by the following result:

Theorem [AMR+19]

Two graphs G, H are quantum isomorphic if and only if there is a unital C^*-algebra $\mathcal{A}$ equipped with a tracial state ρ and projections $\{P_{gh}\}_{g \in V(G), h \in V(H)} \subseteq \mathcal{A}$ such that:

1. $\sum_{h' \in V(H)} P_{gh'} = \sum_{g' \in V(G)} P_{g'h} = 1$ for all $g \in V(G)$ and $h \in V(H)$.
2. The operator matrix $P = [P_{g,h}]_{g \in V(G), h \in V(H)}$ satisfies $A_G P = P A_H$.

In this case, one can construct a perfect strategy p such that

$$p(h, h' \mid g, g') = \rho(P_{gh} P_{g'h'}).$$

The NPA Hierarchy

- The NPA [NPA08] hierarchy is a sequence of SDP relaxations that provide an upper bound on how well quantum strategies perform in a given nonlocal game.

The NPA Hierarchy

- The NPA [NPA08] hierarchy is a sequence of SDP relaxations that provide an upper bound on how well quantum strategies perform in a given nonlocal game.
- More generally, it provides SDP relaxations of any noncommutative polynomial optimization problem, making it a noncommutative analogue of the Lasserre hierarchy.

The NPA Hierarchy

- The NPA [NPA08] hierarchy is a sequence of SDP relaxations that provide an upper bound on how well quantum strategies perform in a given nonlocal game.
- More generally, it provides SDP relaxations of any noncommutative polynomial optimization problem, making it a noncommutative analogue of the Lasserre hierarchy.
- What makes it particularly interesting is that the hierarchy converges, i.e., the limit of the optimal value of the SDP relaxations is equal to the *quantum value* of the game.

The NPA Hierarchy

- The NPA [NPA08] hierarchy is a sequence of SDP relaxations that provide an upper bound on how well quantum strategies perform in a given nonlocal game.
- More generally, it provides SDP relaxations of any noncommutative polynomial optimization problem, making it a noncommutative analogue of the Lasserre hierarchy.
- What makes it particularly interesting is that the hierarchy converges, i.e., the limit of the optimal value of the SDP relaxations is equal to the *quantum value* of the game.
- We present a synchronous version of the NPA hierarchy for the isomorphism game.

First Level of the NPA Hierarchy (Intuition)

- Say we have a perfect strategy for the G, H-isomorphism game. Hence, we have a C^*-algebra $\mathcal{A}$, a tracial state ρ, and projections $\{P_{x,a}\}$ satisfying certain conditions.

First Level of the NPA Hierarchy (Intuition)

- Say we have a perfect strategy for the G, H-isomorphism game. Hence, we have a C^*-algebra $\mathcal{A}$, a tracial state ρ, and projections $\{P_{x,a}\}$ satisfying certain conditions.
- By using the GNS construction, we may assume that $\mathcal{A}$ is represented on a Hilbert space $\mathcal{H}$ by a map $\pi : \mathcal{A} \to \mathcal{B}(H)$ and there is a unit vector $|\psi\rangle \in \mathcal{H}$ such that

$$\rho(a) = \langle \psi \mid \pi(a) \mid \psi \rangle .$$

First Level of the NPA Hierarchy (Intuition)

- Now, consider the set of vectors $|v_{gh}\rangle = \pi(P_{gh})\,|\psi\rangle$ for $g \in V(G)$ and $h \in V(H)$ and let $\mathcal{R}$ denote their gram matrix. Note that all the information for judging whether this strategy is perfect is contained in $\mathcal{R}$. Indeed, we have

$$p(h, h' \mid g, g') = \langle \psi \mid \rho(P_{gh} P_{g'h'}) \mid \psi \rangle = \mathcal{R}_{gh, g'h'}.$$

First Level of the NPA Hierarchy (Intuition)

- Now, consider the set of vectors $|v_{gh}\rangle = \pi(P_{gh})|\psi\rangle$ for $g \in V(G)$ and $h \in V(H)$ and let $\mathcal{R}$ denote their gram matrix. Note that all the information for judging whether this strategy is perfect is contained in $\mathcal{R}$. Indeed, we have

$$p(h, h' \mid g, g') = \langle \psi \mid \rho(P_{gh}P_{g'h'}) \mid \psi \rangle = \mathcal{R}_{gh,g'h'}.$$

- Also note that $\mathcal{R}$ satisfies certain properties. For example, for any $g', g \in V(G)$ we have $\sum_{h,h' \in V(H)} \mathcal{R}_{gh,g'h'} = \sum_{h,h' \in V(H)} \langle \psi \mid \pi(P_{gh}P_{g'h'}) \mid \psi \rangle = \langle \psi \mid \psi \rangle = 1.$

First Level of the NPA Hierarchy (Intuition)

- Now, consider the set of vectors $|v_{gh}\rangle = \pi(P_{gh})\,|\psi\rangle$ for $g \in V(G)$ and $h \in V(H)$ and let $\mathcal{R}$ denote their gram matrix. Note that all the information for judging whether this strategy is perfect is contained in $\mathcal{R}$. Indeed, we have

$$p(h, h' \mid g, g') = \langle \psi \mid \rho(P_{gh}P_{g'h'}) \mid \psi \rangle = \mathcal{R}_{gh,g'h'}.$$

- Also note that $\mathcal{R}$ satisfies certain properties. For example, for any $g', g \in V(G)$ we have $\sum_{h,h' \in V(H)} \mathcal{R}_{gh,g'h'} = \sum_{h,h' \in V(H)} \langle \psi \mid \pi(P_{gh}P_{g'h'}) \mid \psi \rangle = \langle \psi \mid \psi \rangle = 1.$
- These conditions are affine and can be used to give an SDP relaxation of quantum isomorphism, which is the first level of the NPA hierarchy. For higher levels, we consider matrices indexed by longer words in $\Sigma := V(G) \times V(H)$.

A Synchronous NPA Hierarchy for Quantum Isomorphism

For two graphs G, H a *certificate* for the k^{th} level of the NPA hierarchy of the (G, H)-isomorphism game is a positive semidefinite matrix $\mathcal{R} \in M_{\Sigma^{\leq k}}(\mathbb{C})$ such that:

- $\mathcal{R}_{\epsilon,\epsilon} = 1$.
- $\mathcal{R}_{s,t} = \mathcal{R}_{s',t'}$ for all $r, s, r', s' \in \Sigma^{\leq k}$, such that $s^R t \sim (s')^R(t')$, where we define $\sim$ to be the coarsest equivalence relation satisfying the following two properties:
  - For each $x, a \in X \times A$, $s(x,a)(x,a)t \sim s(x,a)t$ for all $s, t \in \Sigma^*$.
  - $st \sim ts$ for all words $s, t \in \Sigma^*$.
- For all words $s, s' \in \Sigma^{\leq k}$, $g \in V(G)$, $h \in V(H)$ such that $s(g,h)s' \in \Sigma^{\leq k}$, one has

$$\sum_{h' \in V(H)} \mathcal{R}_{s(g,h')s',t} = \mathcal{R}_{ss',t} \text{ for all } t \in \Sigma^{\leq k}$$

$$\sum_{g' \in V(G)} \mathcal{R}_{s(g',h)s',t} = \mathcal{R}_{ss',t} \text{ for all } t \in \Sigma^{\leq k}$$

- For all $s, t \in \Sigma^{\leq k}$, if there exist consecutive $gh, g'h' \in \Sigma$ in $s^R t$ such that $\text{rel}(g, g') \neq \text{rel}(h, h')$ then $\mathcal{R}_{s,t} = 0$.

Bilabelled Graphs

Bilabelled Graphs

A (k, l)-bilabelled graph $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ consists of

Bilabelled Graphs

A (k, l)-bilabelled graph $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ consists of

- a graph F,
- a tuple of vertices $\boldsymbol{u} = (u_1, \ldots, u_k) \in V(G)^k$ called the inlabels, and
- a tuple of vertices $\boldsymbol{v} = (v_1, \ldots, v_l) \in V(G)^l$ called the outlabels.

Bilabelled Graphs

A (k, l)-bilabelled graph $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ consists of

- a graph F,
- a tuple of vertices $\boldsymbol{u} = (u_1, \ldots, u_k) \in V(G)^k$ called the inlabels, and
- a tuple of vertices $\boldsymbol{v} = (v_1, \ldots, v_l) \in V(G)^l$ called the outlabels.

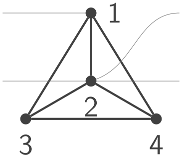


Figure: $\boldsymbol{F} = (K_4, (1,2), (2,2))$

Homomorphism Tensors

Let $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ be a k, l-bilabelled graph and G be a graph. The *homomorphism tensor* of $\boldsymbol{F}$ for G is $\boldsymbol{F}_G \in \mathbb{C}^{V(G)^k \times V(G)^l}$ whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is the number of homomorphisms $h \colon F \to G$ such that $h(\boldsymbol{u}_i) = \boldsymbol{x}_i$ and $h(\boldsymbol{v}_j) = \boldsymbol{y}_j$ for all $i \in [k]$ and $j \in [l]$.

Remark. $\hom(F, G) =$ sum of the entries of $\boldsymbol{F}_G$.

Homomorphism Tensors

Let $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ be a k, l-bilabelled graph and G be a graph. The *homomorphism tensor* of $\boldsymbol{F}$ for G is $\boldsymbol{F}_G \in \mathbb{C}^{V(G)^k \times V(G)^l}$ whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is the number of homomorphisms $h \colon F \to G$ such that $h(\boldsymbol{u}_i) = \boldsymbol{x}_i$ and $h(\boldsymbol{v}_j) = \boldsymbol{y}_j$ for all $i \in [k]$ and $j \in [l]$.
Remark. $\hom(F, G) =$ sum of the entries of $\boldsymbol{F}_G$.

Example. $\boldsymbol{A} = (K_2, (1), (2))$ 

Let $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ be a k, l-bilabelled graph and G be a graph. The *homomorphism tensor* of $\boldsymbol{F}$ for G is $\boldsymbol{F}_G \in \mathbb{C}^{V(G)^k \times V(G)^l}$ whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is the number of homomorphisms $h \colon F \to G$ such that $h(\boldsymbol{u}_i) = \boldsymbol{x}_i$ and $h(\boldsymbol{v}_j) = \boldsymbol{y}_j$ for all $i \in [k]$ and $j \in [l]$.

Remark. $\hom(F, G) =$ sum of the entries of $\boldsymbol{F}_G$.

Example. $\boldsymbol{A} = (K_2, (1), (2))$ 

$$(\boldsymbol{A}_G)_{g,g'} =$$

Let $\boldsymbol{F} = (F, \boldsymbol{u}, \boldsymbol{v})$ be a k, l-bilabelled graph and G be a graph. The *homomorphism tensor* of $\boldsymbol{F}$ for G is $\boldsymbol{F}_G \in \mathbb{C}^{V(G)^k \times V(G)^l}$ whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is the number of homomorphisms $h \colon F \to G$ such that $h(\boldsymbol{u}_i) = \boldsymbol{x}_i$ and $h(\boldsymbol{v}_j) = \boldsymbol{y}_j$ for all $i \in [k]$ and $j \in [l]$.
Remark. $\hom(F, G) =$ sum of the entries of $\boldsymbol{F}_G$.

Example. $\boldsymbol{A} = (K_2, (1), (2))$

$$(\boldsymbol{A}_G)_{g,g'} = \begin{cases} 1 & \text{if } gg' \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Operations on bilabelled graphs: Series composition

Operations on bilabelled graphs: Series composition

Theorem. For a graph G and bilabelled graphs $\boldsymbol{F}, \boldsymbol{F'}$,

$$\boldsymbol{F}_G \boldsymbol{F'}_G = (\boldsymbol{F} \circ \boldsymbol{F'})_G ,$$

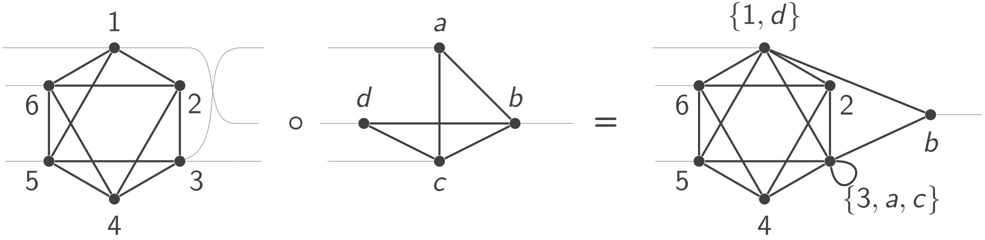
where $\boldsymbol{F} \circ \boldsymbol{F'}$ is defined as:

Operations on bilabelled graphs: Series composition

Theorem. For a graph G and bilabelled graphs $\boldsymbol{F}, \boldsymbol{F}'$,

$$\boldsymbol{F}_G \boldsymbol{F}'_G = (\boldsymbol{F} \circ \boldsymbol{F}')_G \,,$$

where $\boldsymbol{F} \circ \boldsymbol{F}'$ is defined as:

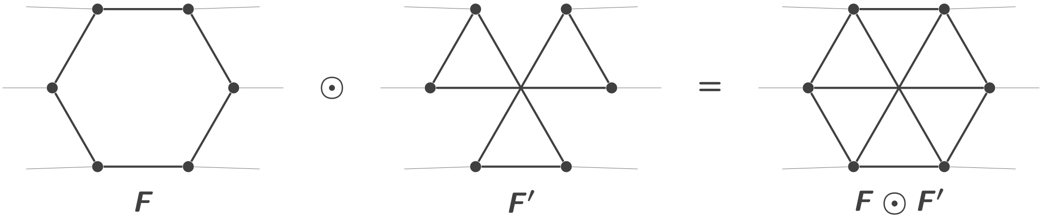


Operations on bilabelled graphs: Parallel composition

Theorem. For a graph G and bilabelled graphs $\boldsymbol{F}, \boldsymbol{F'}$,

$$\boldsymbol{F}_G \odot \boldsymbol{F'}_G = (\boldsymbol{F} \odot \boldsymbol{F'})_G ,$$

where $\boldsymbol{F}_G \odot \boldsymbol{F'}_G$ is the *entrywise product*, and $\boldsymbol{F} \odot \boldsymbol{F'}$ is defined as:

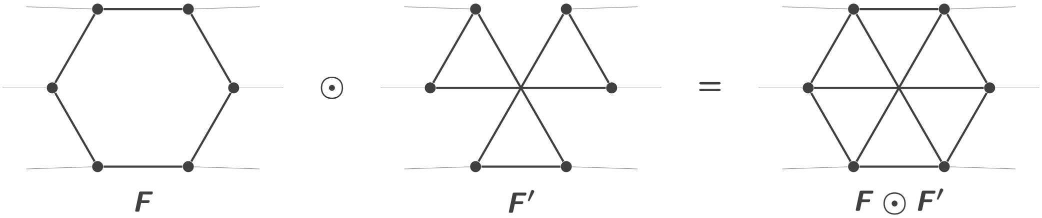


Operations on bilabelled graphs: Parallel composition

Theorem. For a graph G and bilabelled graphs $\boldsymbol{F}, \boldsymbol{F'}$,

$$\boldsymbol{F}_G \odot \boldsymbol{F'}_G = (\boldsymbol{F} \odot \boldsymbol{F'})_G\,,$$

where $\boldsymbol{F}_G \odot \boldsymbol{F'}_G$ is the *entrywise product*, and $\boldsymbol{F} \odot \boldsymbol{F'}$ is defined as:



Other operations: transposition, trace, and cyclic permutations.

Atomic Graphs

Atomic Graphs

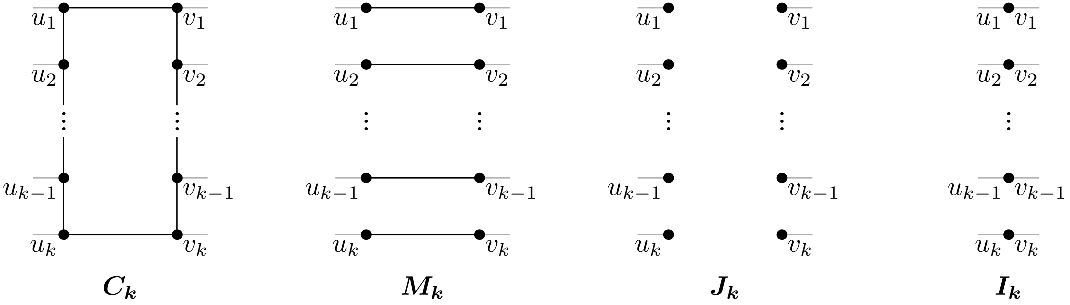


Figure: Atomic Graphs

The class of bilabelled graphs $\mathcal{P}_k$

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\mathcal{Q}_k = \mathcal{Q}_k^P \cup \mathcal{Q}_k^S$.

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\mathcal{Q}_k = \mathcal{Q}_k^P \cup \mathcal{Q}_k^S$.

Then $\mathcal{P}_k$ is the class of (k, k)-bilabelled graphs generated by the elements of $\mathcal{Q}_k$ under

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\mathcal{Q}_k = \mathcal{Q}_k^P \cup \mathcal{Q}_k^S$.

Then $\mathcal{P}_k$ is the class of (k, k)-bilabelled graphs generated by the elements of $\mathcal{Q}_k$ under

- series composition,

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\mathcal{Q}_k = \mathcal{Q}_k^P \cup \mathcal{Q}_k^S$.

Then $\mathcal{P}_k$ is the class of (k, k)-bilabelled graphs generated by the elements of $\mathcal{Q}_k$ under

- series composition,
- parallel composition with the elements of $\mathcal{Q}_k^P$,

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\mathcal{Q}_k = \mathcal{Q}_k^P \cup \mathcal{Q}_k^S$.

Then $\mathcal{P}_k$ is the class of (k, k)-bilabelled graphs generated by the elements of $\mathcal{Q}_k$ under

- series composition,
- parallel composition with the elements of $\mathcal{Q}_k^P$,
- transpose,

The class of bilabelled graphs $\mathcal{P}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\mathcal{Q}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\mathcal{Q}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\mathcal{Q}_k = \mathcal{Q}_k^P \cup \mathcal{Q}_k^S$.

Then $\mathcal{P}_k$ is the class of (k, k)-bilabelled graphs generated by the elements of $\mathcal{Q}_k$ under

- series composition,
- parallel composition with the elements of $\mathcal{Q}_k^P$,
- transpose,
- and cyclic permutations.

The class of bilabelled graphs $\boldsymbol{\mathcal{P}}_k$

Definition. For $k \in \mathbb{N}$, define

1. $\boldsymbol{\mathcal{Q}}_k^P$ to be the set of all minors of $\boldsymbol{C}_k$;
2. $\boldsymbol{\mathcal{Q}}_k^S$ to be the set of all minors of $\boldsymbol{M}_k$;
3. $\boldsymbol{\mathcal{Q}}_k = \boldsymbol{\mathcal{Q}}_k^P \cup \boldsymbol{\mathcal{Q}}_k^S$.

Then $\boldsymbol{\mathcal{P}}_k$ is the class of (k, k)-bilabelled graphs generated by the elements of $\boldsymbol{\mathcal{Q}}_k$ under

- series composition,
- parallel composition with the elements of $\boldsymbol{\mathcal{Q}}_k^P$,
- transpose,
- and cyclic permutations.

Definition. $\mathcal{P}_k = \{F \ : \ \exists(F, \boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{\mathcal{P}}_k\}$

Structure of the Proof

The k^{th} level of the NPA hierarchy for the (G, H)-isomorphism game is feasible

$\Updownarrow$

There is a level-k quantum isomorphism map from G to H

$\Updownarrow$

G and H are homomorphism indistinguishable over $\mathcal{P}_k$

Convergence of the NPA Hierarchy

Convergence of the NPA Hierarchy

- Since the NPA hierarchy converges, we have the following:

Convergence of the NPA Hierarchy

- Since the NPA hierarchy converges, we have the following:

Lemma [KRSZ26]

The following are equivalent:

1. $G \cong_q H$
2. The k^{th}-level of the NPA hierarchy is feasible for all $k \in \mathbb{N}$.

Convergence of the NPA Hierarchy

- Since the NPA hierarchy converges, we have the following:

Lemma [KRSZ26]

The following are equivalent:

1. $G \cong_q H$
2. The k^{th}-level of the NPA hierarchy is feasible for all $k \in \mathbb{N}$.

- As a consequence of our main result, we may write the following:

Convergence of the NPA Hierarchy

- Since the NPA hierarchy converges, we have the following:

Lemma [KRSZ26]

The following are equivalent:

1. $G \cong_q H$
2. The k^{th}-level of the NPA hierarchy is feasible for all $k \in \mathbb{N}$.

- As a consequence of our main result, we may write the following:

Corollary [KRSZ26]

The following are equivalent:

1. $G \cong_q H$
2. $G \cong_{\mathcal{P}} H$, where $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$.

Homomorphism Indistinguishability over Planar Graphs

Homomorphism Indistinguishability over Planar Graphs

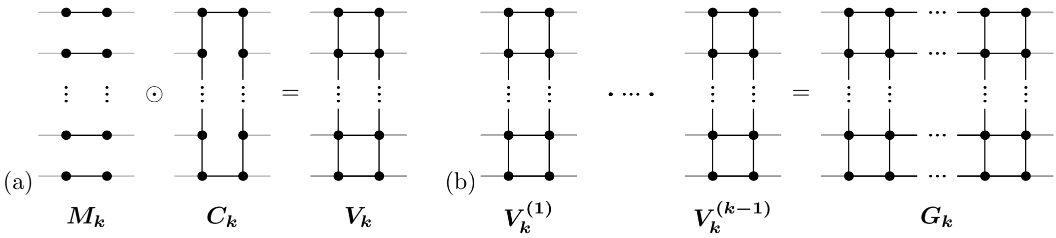
- One can easily show that each $\mathcal{P}_k$ contains only of planar graphs. Hence, to establish Mančinska and Roberson's result, we need to show that $\mathcal{P}$ contains all planar graphs.

Homomorphism Indistinguishability over Planar Graphs

- One can easily show that each $\mathcal{P}_k$ contains only of planar graphs. Hence, to establish Mančinska and Roberson's result, we need to show that $\mathcal{P}$ contains all planar graphs.
- Since each $\mathcal{P}_k$ is minor-closed, it suffices to show that $\mathcal{P}$ contains arbitrarily large grids.

Homomorphism Indistinguishability over Planar Graphs

- One can easily show that each $\mathcal{P}_k$ contains only of planar graphs. Hence, to establish Mančinska and Roberson's result, we need to show that $\mathcal{P}$ contains all planar graphs.
- Since each $\mathcal{P}_k$ is minor-closed, it suffices to show that $\mathcal{P}$ contains arbitrarily large grids.



Homomorphism Indistinguishability over Graph Classes of Bounded Treewidth

Homomorphism Indistinguishability over Graph Classes of Bounded Treewidth

- *Treewidth* is a parameter that determines how close a graph is to being a tree. For example, forests have treewidth 1, series-parallel graphs have treewidth 2, and the $k \times k$ grid has treewidth k.

Homomorphism Indistinguishability over Graph Classes of Bounded Treewidth

- *Treewidth* is a parameter that determines how close a graph is to being a tree. For example, forests have treewidth 1, series-parallel graphs have treewidth 2, and the $k \times k$ grid has treewidth k.
- In [Sep24], Seppelt showed that homomorphism indistinguishability over minor-closed graph classes of bounded treewidth can be decided in polynomial time by a randomized algorithm.

Randomized Polynomial Time Algorithm for the NPA hierarchy

- Each level of the NPA hierarchy, being an SDP, can be solved in polynomial time to arbitrary precision, but not exactly.

Randomized Polynomial Time Algorithm for the NPA hierarchy

- Each level of the NPA hierarchy, being an SDP, can be solved in polynomial time to arbitrary precision, but not exactly.
- The graph classes $\mathcal{P}_k$ that we construct are minor-closed and have treewidth at most $3k-1$. Hence, it follows that there is a randomzided polynomial time algorithm deciding exact feasibility of each level of the NPA hierarchy for the graph isomorphism.

Randomized Polynomial Time Algorithm for the NPA hierarchy

- Each level of the NPA hierarchy, being an SDP, can be solved in polynomial time to arbitrary precision, but not exactly.
- The graph classes $\mathcal{P}_k$ that we construct are minor-closed and have treewidth at most $3k-1$. Hence, it follows that there is a randomzided polynomial time algorithm deciding exact feasibility of each level of the NPA hierarchy for the graph isomorphism.
- We improve upon this result by making the dependence on k *explicit*:

Theorem [KRSZ26]

There exists a randomized algorithm which decides, given graphs G and H and an integer $k \geq 1$, whether the k^{th}-level of the NPA hierarchy for the (G, H)-isomorphism game is feasible. The algorithm always runs in time $n^{O(k)} k^{O(1)}$ for $n := \max\{|V(G)|, |V(H)|\}$, accepts all YES-instances, and accepts NO-instances with probability less than one half.

Thank You!

References I

Albert Atserias, Laura Mančinska, David E. Roberson, Robert Šámal, Simone Severini, and Antonios Varvitsiotis.
Quantum and non-signalling graph isomorphisms.
Journal of Combinatorial Theory, Series B, 136:289–328, 2019.

Prem Nigam Kar, David E Roberson, Tim Seppelt, and Peter Zeman.
Npa hierarchy for quantum isomorphism and homomorphism indistinguishability.
Quantum, 10:1989, 2026.

László Lovász.
Operations with structures.
Acta Mathematica Academiae Scientiarum Hungarica, 18(3):321–328, September 1967.

References II

Laura Mančinska and David E. Roberson.
Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs.
In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 661–672, 2020.

Miguel Navascués, Stefano Pironio, and Antonio Acín.
A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations.
New Journal of Physics, 10(7):073013, jul 2008.

Tim Seppelt.
An algorithmic meta theorem for homomorphism indistinguishability.
In *49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024)*, pages 82–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2024.