Three-qubit nonlocality paradoxes: beyond GHZ

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June 6, 2026

Strong contextuality

Strong contextuality is an extremal form of contextuality witnessed by a *logical paradox*: no hidden variable is consistent with the possible observations.

$$|GHZ\rangle \propto |000\rangle + |111\rangle$$

A	B	C	000	001	010	011	100	101	110	111
X_A	X_B	X_C	1/4	0	0	1/4	0	1/4	1/4	0
X_A	Y_B	Y_C	0	1/4	1/4	0	1/4	0	0	1/4
Y_A	X_B	Y_C	0	1/4	1/4	0	1/4	0	0	1/4
Y_A	Y_B	X_C	0	1/4	1/4	0	1/4	0	0	1/4

$$x_A \oplus x_B \oplus x_C = 0$$
$$x_A \oplus y_B \oplus y_C = 1$$
$$y_A \oplus x_B \oplus y_C = 1$$
$$y_A \oplus y_B \oplus x_C = 1$$

Nonlocality paradoxes

$(\mathcal{M}, \mathcal{C}, \mathcal{O})$: measurements in $\mathcal{M}$ with outcomes $\mathcal{O}$; compatibility captured by $\mathcal{C} \subset \mathcal{P}(\mathcal{M})$

Nonlocality (Bell) experiments: $\mathcal{M} = \bigsqcup_i M_i$ and $\mathcal{C} = \prod_i M_i$

Empirical model: distributions $P_C : \mathcal{O}^C \to [0,1]$ for all $C \in \mathcal{C}$, satisfying nonsignalling

Hidden variable/global assignment: $g : \mathcal{M} \to \mathcal{O}$

Hidden variable model: a distribution μ over $\mathcal{O}^{\mathcal{M}}$ recovering an empirical model as marginals

Nonlocality: an empirical model for a Bell experiment with no hidden variable model

Strong contextuality/nonlocality paradox: an empirical model $\{P_C\}_{C \in \mathcal{C}}$ such that for every hidden variable $g \in \mathcal{O}^{\mathcal{M}}$, there exists $C \in \mathcal{C}$ such that $P_C(g|_C) = 0$

Every hidden variable predicts an empirically impossible event $\implies$ no hidden variable model

Constraint satisfaction

Literals: (M, o) means $M = o$ in a CSP with variables $\mathcal{M}$ and domain $\mathcal{O}$

Event terms: For $e \in \mathcal{O}^C$, define $\gamma(e) := \bigwedge_{M \in C}(M, e(M))$.

Given $\{P_C\}_{C \in \mathcal{C}}$, define the support relation and possible events term of $C \in \mathcal{C}$:

$$\mathsf{supp}_C := \{e \in \mathcal{O}^C : P_C(e) \neq 0\} \qquad \mathsf{Poss}(C) := \bigvee_{e \in \mathsf{supp}_C} \gamma(e)$$

An empirical model is a (strong) contextuality/nonlocality paradox iff

$$\{(C, \mathsf{supp}_C) : C \in \mathcal{C}\} \text{ is unsatisfiable} \iff \bigwedge_{C \in \mathcal{C}} \mathsf{Poss}(C) \text{ is unsatisfiable.}$$

A quantum nonlocality paradox: choice of n-partite quantum state, measurement sets.

A minimal paradox: removing any measurement $\implies$ no more paradox.

Significance

Maximally sharp certificates of nonclassical behaviour.

Contextual fraction: All empirical models are a convex combination of a strongly contextual/nonlocal one and a classically random one.

Success probability for nonlocal games and measurement-based computation (Raussendorf).

Resources for cryptography: e.g. quantum secret sharing, randomness amplification, expansion, and certification.

Key technical tools in establishing unconditional quantum advantage, e.g. BGK separation of shallow quantum and classical circuit complexity. See also $\mathrm{MIP}^* = \mathrm{RE}$.

Summary of results

Complete structural classification of biconditioned parity proofs of three-qubit nonlocality paradoxes.

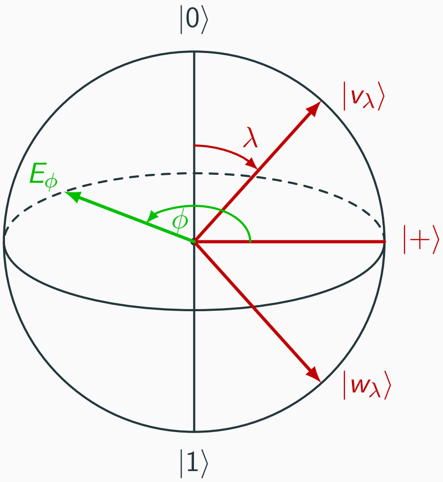
Canonical combinatorial description with easily verifiable necessary and sufficient conditions.

The delicate interaction of measurements to produce the necessary interference for impossible events imposes symmetry and structure...

Nonlocality paradoxes are incredibly abundant and can exhibit unexpected features.

Prior work

Balanced states and equatorial measurements (Abramsky et al., TQC '17)



$\lambda \in [0, \frac{\pi}{2})$
$\phi \in [0, \pi)$

$$|v_\lambda\rangle = \cos\frac{\lambda}{2}|0\rangle + \sin\frac{\lambda}{2}|1\rangle$$

$$|w_\lambda\rangle = \sin\frac{\lambda}{2}|0\rangle + \cos\frac{\lambda}{2}|1\rangle$$

$$E_\phi = \cos(\phi)X + \sin(\phi)Y$$

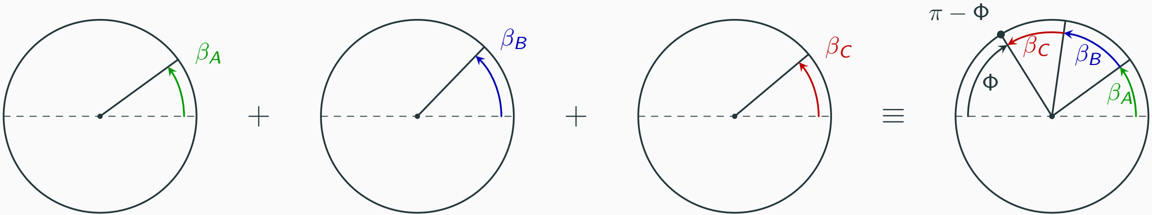
$$|B(\boldsymbol{\lambda}, \Phi)\rangle \propto |v_{\lambda_1}\rangle|v_{\lambda_2}\rangle|v_{\lambda_3}\rangle + e^{i\Phi}|w_{\lambda_1}\rangle|w_{\lambda_2}\rangle|w_{\lambda_3}\rangle$$

Amplitudes to logic (Abramsky et al., TQC ’17)

Impossible events are exactly the zeros of the amplitude created by interference.

$$(A, B, C) \mapsto (a, b, c) \text{ impossible}$$
$$\iff \beta(\lambda_1, A + a\pi) + \beta(\lambda_2, B + b\pi) + \beta(\lambda_3, C + c\pi) \equiv \pi - \Phi$$

$$\beta(\lambda, \phi) := \phi - 2\arctan\left(\frac{\cos(\lambda/2)\sin\phi}{\sin(\lambda/2) + \cos(\lambda/2)\cos\phi}\right)$$



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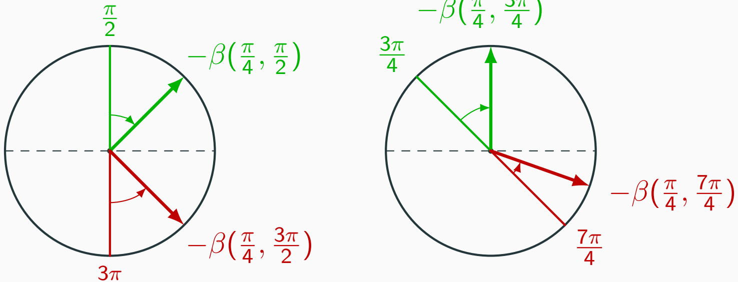


Sequence of biconditioned parity proofs (Abramsky et al., TQC '17)

Biconditional parity proof $\iff |M_3| = 2$ and $\lambda_1 = \lambda_2 = 0$

For even N:

$$|B((0,0,\frac{\pi}{2}-\frac{\pi}{N}),0)\rangle, \qquad M_A = M_B = \frac{\pi}{N}\mathbb{Z}_N, \qquad M_C = \{0, \frac{\pi}{2}\} = \{X, Y\}$$

$$P := \bigoplus_{i\in\mathbb{Z}_N} a_i \oplus \bigoplus_{j\in\mathbb{Z}_N} b_j.$$

$$\begin{aligned} &\text{Charlie measures } X && \implies && P = 1 \\ &\text{Charlie measures } Y,\ c_Y = 0 && \implies && P = 0 \\ &\text{Charlie measures } Y,\ c_Y = 1 && \implies && P = 0 \end{aligned}$$

Formalism

Charlie Conditionings

Charlie's measurements: $M_3 = \{C_0, \ldots, C_{n-1}\}$

Charlie conditioning: $(C_m, z_m) \in M_3 \times \mathbb{Z}_2$

Charlie tick: $T_{m,z_m} := \beta(\lambda_3, C_m + z_m\pi)$

Total Charlie assignment: $\mathbf{z} = (z_0, \ldots, z_{n-1}) \in \mathcal{O}^{M_3} = \mathbb{Z}_2^n$

Conditioned impossibility when $\lambda_1 = \lambda_2 = 0$:

$$\beta(\lambda_1, A + a\pi) + \beta(\lambda_2, B + b\pi) + \beta(\lambda_3, C + c\pi) \equiv \pi - \Phi$$

$$A + B \equiv T_{C,c} + (a \oplus b \oplus 1)\pi$$

Charlie conditioning $\Longrightarrow$ partial matching on M_1, M_2

Forbidden events and branch formulae

$$E_{m,z} := \left\{ \big((A,a),(B,b)\big) : (A,B,C_m) \to (a,b,z) \text{ impossible} \right\} \subset (M_1 \times \mathcal{O}) \times (M_2 \times \mathcal{O})$$

$$\Omega_m(z) := \bigwedge_{\big((A,a),(B,b)\big) \in E_{m,z}} (A, a \oplus 1) \vee (B, b \oplus 1)$$

$$\Omega(\mathbf{z}) := \bigwedge_{m=0}^{n-1} \Omega_m(z_m)$$

$$\bigwedge_{C \in \mathcal{C}} \mathrm{Poss}(C) \equiv \bigvee_{\mathbf{z} \in \mathbb{Z}_2^n} \left[\left(\bigwedge_{m=0}^{n-1} (C_m, z_m) \right) \wedge \Omega(\mathbf{z}) \right]$$

strong nonlocality paradox $\iff$ $\forall \mathbf{z} \in \mathbb{Z}_2^n,\ \Omega(\mathbf{z})$ is unsatisfiable

2-SAT implication graph of a Charlie conditioning

Fix a Charlie conditioning $(C_m, z_m) \in M_3 \times \mathbb{Z}_2$

Vertices: Alice and Bob literals.

Directed edges from forbidden pairs.

$$((A,a),(B,b)) \in E_{m,z} \quad \Longrightarrow \quad \begin{cases} (A,a) \to (B, b \oplus 1) \\ (B,b) \to (A, a \oplus 1). \end{cases}$$

All edges are bidirected for parity proofs giving a bipartite graph

$$((A,a),(B,b)) \in E_{m,z} \iff ((A, a \oplus 1),(B, b \oplus 1)) \in E_{m,z}.$$

$$\boxed{\Omega(\mathbf{z}) \text{ is unsatisfiable } \iff \exists M \in M_1 \sqcup M_2, o \in \mathbb{Z}_2 : \ (M, o) \rightsquigarrow (M, o \oplus 1)}$$

Witness paths force rational tick differences

A witness path in $\Omega(\mathbf{z})$ alternates between the two Charlie conditionings:

$$(A_0, a_0) \xrightarrow{(C_0,z_0)} (B_0, b_0) \xrightarrow{(C_1,z_1)} (A_1, a_1) \xrightarrow{(C_0,z_0)} \cdots$$

Ignoring outcomes, the two consecutive impossibility equations are

$$A_q + B_q \equiv T_{0,z_0} \pmod{\pi}, \qquad A_{q+1} + B_q \equiv T_{1,z_1} \pmod{\pi}.$$

Subtracting gives the Alice-to-Alice step

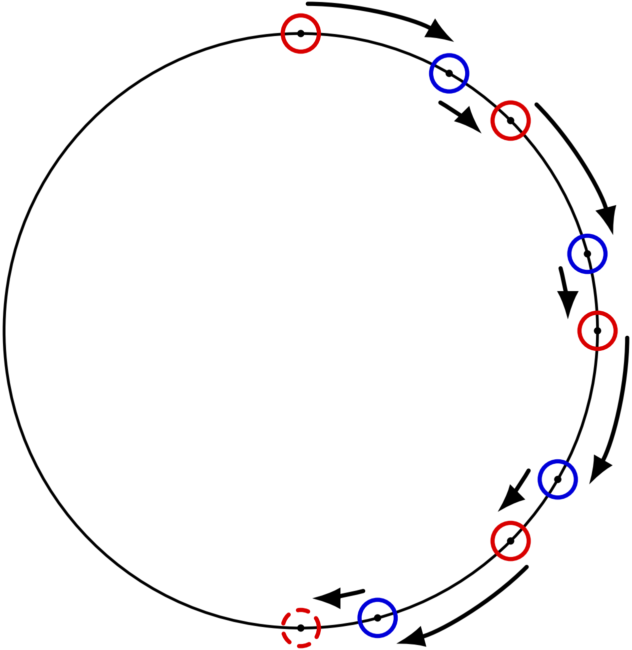
$$A_{q+1} \equiv A_q + \big(T_{1,z_1} - T_{0,z_0}\big) \pmod{\pi}.$$

A witness path ends at the opposite literal of the same measurement:

$$(A_0, a_0) \rightsquigarrow (A_0, a_0 \oplus 1).$$

Hence, for some path length L,

$$L\big(T_{1,z_1} - T_{0,z_0}\big) \equiv 0 \pmod{\pi}.$$



Classification of BPP paradoxes

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We reduce BPPs to a small number of parameters that obey simple, easily-checkable conditions.

minimal BPP = valid Charlie clock + canonical Alice–Bob completion + layer shifts

$$\mathrm{BPP} \cong \mathrm{CanonicalTriples}$$

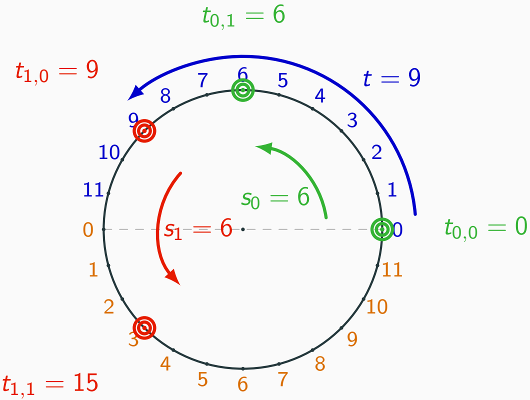
Charlie fixes a clock group $\mathbb{Z}_N$

Alice's measurements: a coset of $\mathbb{Z}_N$ for each total Charlie asssignment

Bob's measurements are determined by Alice's

Additive shifts arrange layers of measurements cosets.

Four ticks on a finite clock



Witness paths force all tick differences to be rational multiples of π.

$$T_{\ell,z} = \frac{\pi}{N}(t_{\ell,z} + \mu), \qquad t_{\ell,z} \in \mathbb{Z}_{2N}.$$

We normalise the four tick indices as

$$t_{0,0} = 0, \qquad t_{0,1} = s_0,$$
$$t_{1,0} = t, \quad t_{1,1} = t + s_1.$$

A Charlie clock is the data:

$$(N, t, s_0, s_1, \mu)$$

Realisable Charlie clocks

A Charlie clock is realisable when its four ticks come from one interpolant state and two Charlie measurements:

$$T_{\ell,z} = \beta(\lambda, C_\ell + z\pi) \qquad (\ell, z \in \mathbb{Z}_2).$$

$$\boxed{\text{valid clock} = \text{realisable clock} + \text{paradoxical clock}}$$

The realisable clocks fall into four explicit families:

GHZ non-GHZ with X equal spread unequal spread.

Example: equal spread clocks

$$s_0 = s_1 = s, \qquad 0 < t < s < N, \qquad \mu \equiv -\frac{t+s}{2} \pmod{2N}.$$

A total Charlie assignment gives a coset

Fix a total Charlie assignment $\mathbf{z} = (z_0, z_1) \in \mathbb{Z}_2^2$. Define

$$H_{\mathbf{z}} := t_{1,z_1} - t_{0,z_0}, \qquad d_{\mathbf{z}} := \gcd(N, H_{\mathbf{z}}), \qquad D_{\mathbf{z}} := d_{\mathbf{z}}\mathbb{Z}_N,$$
$$u_{\mathbf{z}} := t_{0,z_0}.$$

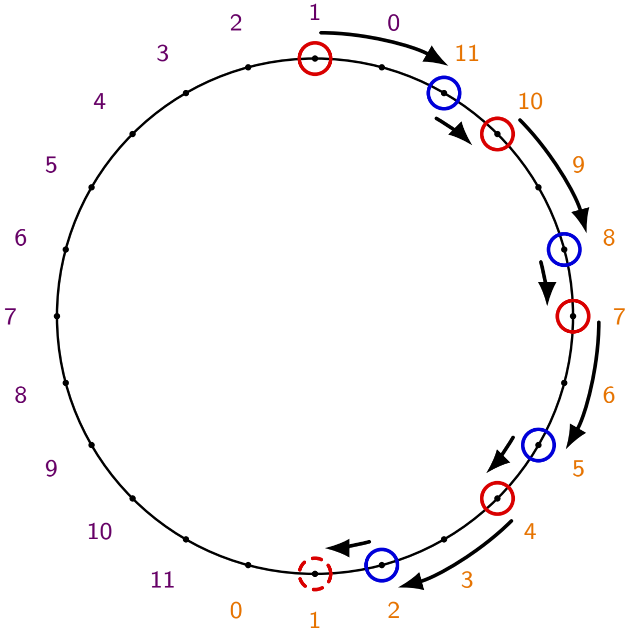
Two implication edges move Alice by

$$j \longmapsto j + H_{\mathbf{z}}.$$

Hence a $\mathbf{z}$-witness path is encoded by paired cosets

$$\text{Alice: } y + D_{\mathbf{z}}, \qquad \text{Bob: } u_{\mathbf{z}} - y + D_{\mathbf{z}}.$$

Once all Charlie's outcomes are fixed, witnesses are cosets.


1
2
3
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7
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9
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0
0
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8
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6
5
4
3
2
1

Paradoxicality is an odd round trip

Iterated rotations by $H_{\mathbf{z}}$ reaches the opposite literal exactly when

$$\boxed{\frac{H_{\mathbf{z}}}{d_{\mathbf{z}}} \text{ is odd.}}$$

Thus a Charlie clock is paradoxical when this oddness condition holds for every $\mathbf{z} \in \mathbb{Z}_2^2$.

Alice and Bob live in clock layers

Once the clock is fixed, Alice and Bob measurements decompose into independent layers:

$$A_{i,j} = \frac{\pi}{N}(j + \alpha_i), \qquad B_{i,k} = \frac{\pi}{N}(k + \mu - \alpha_i),$$

where $j, k \in \mathbb{Z}_N$.

Matched measurements must have the same layer shift α_i. Thus

$$M_1 = \bigsqcup_{i \in I} \frac{\pi}{N}(A_i + \alpha_i), \qquad M_2 = \bigsqcup_{i \in I} \frac{\pi}{N}(B_i + \mu - \alpha_i),$$

$$A_i, B_i \subseteq \mathbb{Z}_N.$$

Edges do not mix layers.

Alice–Bob completions

Once the clock is valid, Alice and Bob complete it by choosing cosets.

For selected total Charlie assignments $S \subseteq \mathbb{Z}_2^2$, choose

$$\iota : S \to I, \qquad y_{\mathbf{z}} \in \mathbb{Z}_N / D_{\mathbf{z}} \quad (\mathbf{z} \in S).$$

Each selected $\mathbf{z}$ contributes

$$\text{Alice: } y_{\mathbf{z}} + D_{\mathbf{z}}, \qquad \text{Bob: } u_{\mathbf{z}} - y_{\mathbf{z}} + D_{\mathbf{z}}.$$

Some total Charlie assignments need not be selected explicitly: their witness cosets may already be forced by the chosen ones.

This is detected by the shadow test, an elementary congruence/divisibility check:

$$d_{\mathbf{z}} \mid 2d_{\mathbf{w}}, \qquad y_{\mathbf{z}} + p(u_{\mathbf{w}} - u_{\mathbf{z}}) \equiv r + \varepsilon d_{\mathbf{w}} \pmod{d_{\mathbf{z}}}.$$

Beyond BPP

Examples of more exotic paradoxes

- **Non-GHZ GHZ-state-paradox**

$$|GHZ\rangle, \qquad (|M_1|, |M_2|, |M_3|) = (3, 3, 3)$$

$$M_i = \left\{ \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \right\}$$

  not maximally impossible; six parity equations sum to $0 = 1$

- **Alice-Bob parity paradoxes with > 2 Charlie measurements**

$$(4, 4, 3), \qquad (N, N, 3), \qquad (N, N/2, 3)$$

- **Four-Charlie numerical examples**

$$(4, 4, 4), \quad (6, 6, 4), \quad (6, 2, 4), \quad (7, 5, 4)$$

Non-BPP counterexample

$$\rho = \sqrt{2} - 1, \qquad \alpha = \arcsin \rho$$

$$|B((\alpha, \alpha, \pi/6), \pi)\rangle = \frac{|v_\alpha v_\alpha v_{\pi/6}\rangle - |w_\alpha w_\alpha w_{\pi/6}\rangle}{\sqrt{2(1-\rho^2/2)}}$$

$$(|M_1|, |M_2|, |M_3|) = (11, 13, 2), \qquad M_3 = \left\{0, \frac{\pi}{2}\right\}$$

implication-cycle proof, not affine-clock/BPP