The category of quantum graphs is closed

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Quantum graphs of homomorphisms

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- Define a category of quantum graphs;
- Describe both meanings of quantum graph homomorphisms in this category;
- Show that the category is monoidal closed;
- Use the monoidal closure to characterize winning quantum strategies of graph homomorphism games.

Classical graphs

Definition

- A graph is a pair $G = (V_G, e_G)$ consisting of:
  - a set V_G of vertices;
  - a symmetric relation e_G on V_G of edges (write $g_1 \sim g_2 \iff (g_1, g_2) \in e_G$);
- A homomorphism $G \to H$ is a map $\varphi : V_G \to V_H$ such that

$$g_1 \sim g_2 \implies \varphi(g_1) \sim \varphi(g_2).$$

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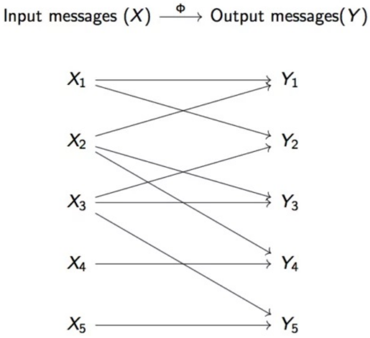
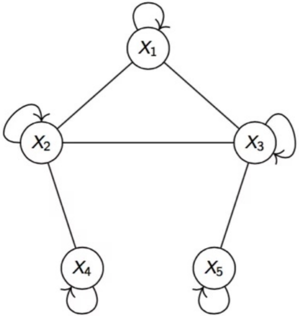
Remarks

- Graphs are allowed to have loops, but are possibly loopless.
- G has loops on every vertex $\iff$ e_G is reflexive.
- G has no loops at all $\iff$ e_G is irreflexive.

Confusibility graphs





Quantum confusibility graphs

Definition (Duan, Severini, Winter)

Let $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a quantum channel (a cptp map) with Kraus matrices $(K_i)_{i\in I}$.
Then the quantum confusibility graph of φ is the subspace

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Observations

- $\mathcal{G}_\varphi$ is independent of the choice of Kraus matrices;
- For pure states p_x and p_y (for unit vectors $x, y \in \mathbb{C}^m$), the following are equivalent:
  - φ confuses inputs p_x and p_y;
  - $\mathrm{tr}(\varphi(p_x)\varphi(p_y)) \neq 0$;
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Definition

A finite (reflexive) quantum graph is a (unital) selfadjoint subspace of $M_m(\mathbb{C})$.

Graph homomorphism games

Definition

Let G and H be finite graphs. The (G, H)-homomorphism game is played by:

1. A verifier who selects a pair of vertices $g_1, g_2 \in V_G$;
2. Alice and Bob, who selects a vertex $h_1 \in V_H$ and a vertex $h_2 \in V_H$, respectively;
3. the winning condition for Alice and Bob is $(g_1 = g_2 \Rightarrow h_1 = h_2)$ & $(g_1 \sim g_2 \Rightarrow h_1 \sim h_2)$.

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Theorem

For finite simple graphs G and H the following are equivalent:

- *There is a classical winning strategy for the (G, H)-homomorphism game;*
- *There is a homomorphism $G \to H$.*

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Theorem

A homomorphism $\varphi : G \to H$ exists $\iff [G, H]_\Box \neq \emptyset$.

Quantum strategies

Definition (informal)

A quantum strategy is:

- Alice and Bob share an entangled quantum state in advance,
- Alice measures it with outcome h_1, choosing her measurement based on g_1;
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Winning quantum strategies

- Found by Mančinska, Roberson (2016), called ‘quantum homomorphism’;
- Can exist even when there is no winning classical strategy.
- Quantum homomorphisms formalized by Musto, Reutter, and Verdon (2018).

Quantum sets

Definition

- A quantum set $\mathcal{X}$ is a collection of Hilbert spaces (called atoms) of finite dimension > 0;
- $\ell^\infty(\mathcal{X}) := \bigoplus_{X \in \mathcal{X}}^{\infty} B(X)$ is its associated hereditarily atomic von Neumann algebra;

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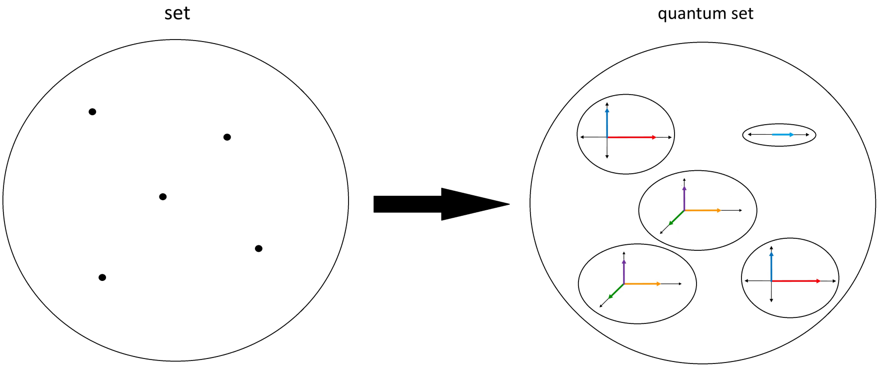
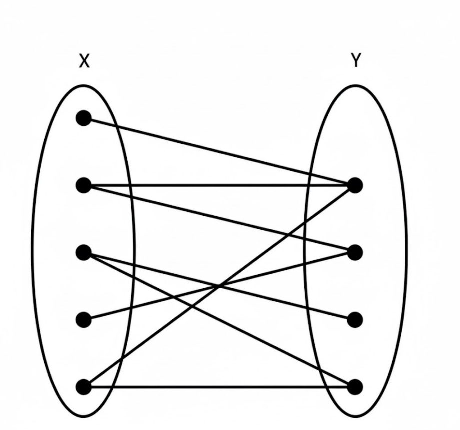


Binary relations

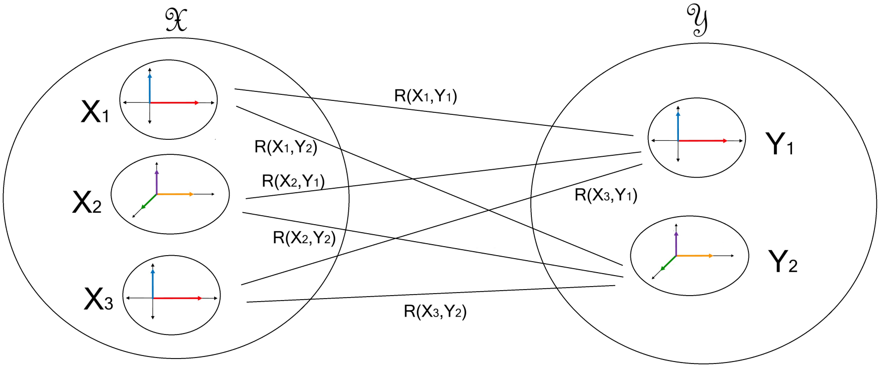
A binary relation $r : X \to Y$ is a bipartite graph (possibly infinite vertices).



Relations between quantum sets

Definition

A relation $R : \mathcal{X} \to \mathcal{Y}$ between quantum sets is a complete bipartite graph, where the edge between atoms $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ is labeled by a space $R(X, Y)$ of linear operators $X \to Y$. We denote the category of quantum sets and relations by **qRel**.



Connections with ordinary sets and von Neumann algebras

- Fully faithful functor $\ell^\infty : \mathbf{qRel} \to \mathbf{WRel}$, the category of von Neumann algebras and Weaver's quantum relations;
- Fully faithful functor $`(-) : \mathbf{Rel} \to \mathbf{qRel}$, $S \mapsto (\mathbb{C} : s \in S)$.

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  - same constraints.

Example: functions between quantum sets

Observation

A function $f : X \to Y$ is a morphism in **Rel** such that

$$f^\dagger \circ f \geq \mathrm{id}_X, \qquad f \circ f^\dagger \leq \mathrm{id}_Y.$$

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Definition

A function $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets is a relation such that

$$F^\dagger \circ F \geq I_\mathcal{X}, \qquad F \circ F^\dagger \leq I_\mathcal{Y}.$$

We denote the category of quantum sets and functions by **qSet**.

Properties of **qSet**

Theorem (Kornell (2020))

- $\ell^\infty : \mathbf{qRel} \to \mathbf{WRel} \xrightarrow{\textit{restricts and corestricts}}$ *fully faithful* $\ell^\infty : \mathbf{qSet} \to \mathbf{WStar}^{\mathrm{op}}$, $\mathbf{WStar}$ = *von Neumann algebras and normal unital $*$-homomorphisms.*
- $`(-) : \mathbf{Rel} \to \mathbf{qRel} \xrightarrow{\textit{restricts and corestricts}}$ *fully faithful* $`(-) : \mathbf{Set} \to \mathbf{qSet}$.
- **qSet** *is generated by the quantum sets* $\mathcal{Q}_n = \{\mathbb{C}^n\}$.
- **qSet** *is symmetric monoidal closed:*
  - *product* $\mathcal{X} \times \mathcal{Y} = \{X \otimes Y : X \in \mathcal{X}, Y \in \mathcal{Y}\}$ *is semicartesian;*
  - *Monoidal unit:* $\mathcal{Q}_1$*;*
  - *Monoidal closure* $\mathcal{Y}^{\mathcal{X}}$ *constructed in terms of representations.*

Discrete quantization is a unifying framework:

- Quantum graphs[1] [2] → quantum error correction, nonlocal games;
- Quantum posets[3] [4]
  - Quantum cpos[4] → recursive pure quantum computing;
  - Quantum suplattices → quantum model of full linear logic?
    → quantum model of differential logic?
- Quantum metric spaces[5] [6] → quantum error correction;

[1] Duan, Severini, Winter (2013)
[2] Mančinska, Roberson (2016)
[3] Weaver (2010)
[4] Kornell, L., Mislove (2026)
[5] Kuperberg, Weaver (2010)
[6] Weaver (2017)

Quantum graphs

Definition of the category Gph of graphs and homomorphisms in relational terms:

- Objects: pairs $G = (V_G, e_G)$ with a set V_G and a relation e_G on V_G such that $e_G^\dagger = e_G$.
- Homomorphisms $\varphi : G \to H$ are maps $V_G \to V_H$ such that $\varphi \circ e_G \leq e_H \circ \varphi$.

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Example (Quantum confusibility graphs)

Let $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a quantum channel. Then $G_\varphi = (\mathcal{Q}_m, E_\varphi)$ with $E_\varphi(\mathbb{C}^m, \mathbb{C}^m) = \mathcal{G}_\varphi$.

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Theorem

There is a fully faithful functor $`(-) : \mathbf{Gph} \to \mathbf{qGph}$, $(V, e) \mapsto (`V, `e)$.

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Observation

There is a bijection between:

- Families of functions $X \to Y$ indexed by a set Z;
- Functions $X \times Z \to Y$.

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Definition

An n-dimensional quantum function $\mathcal{X} \to \mathcal{Y}$ is a quantum family of functions $\mathcal{X} \to \mathcal{Y}$ indexed by $\mathcal{Q}_n$.

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Monoidal products on **Gph** in relational terms

Direct product $G \times H$:

- $V_{G\times H} = V_G \times V_H$;
- $e_{G\times H} = e_G \times e_H$;
- Monoidal unit: $\overline{K}_1$ (one vertex, one loop);

Box product $G \,\square\, H$:

- $V_{G\square H} = V_G \times V_H$;
- $e_{G\square H} = (e_G \times \mathrm{id}_H) \vee (\mathrm{id}_{\mathrm{G}} \times e_H)$;
- Monoidal unit: K_1 (one vertex, no loops).

Monoidal products on **qGph**

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Quantum homomorphisms

Let $Q_n = (\mathcal{Q}_n, 0)$ and $\overline{Q}_n = (\mathcal{Q}_n, I_{\mathcal{Q}_n})$.

Definition

An n-dimensional quantum homomorphism $G \to H$ between (quantum) graphs is an n-dimensional quantum function $\Phi : \mathcal{V}_G \to \mathcal{V}_H$ that satisfies the following equivalent conditions:

- $\Phi : G \times \overline{Q}_n \to H$ is a homomorphism;
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Observations

- These are precisely the quantum homomorphisms of Musto, Reutter, and Verdon (2018);
- Hence, these quantum homomorphisms correspond to winning quantum strategies;
- 1-dimensional quantum homomorphisms = homomorphisms.

The internal hom of **qGph**

Theorem (Kornell, L.)

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Proof (sketch).

Exploit that **qSet** is closed.

1. $\mathcal{V}_{[G,H]}$ is the largest subset $\mathcal{W}$ of the quantum function set $(\mathcal{V}_H)^{\mathcal{V}_G}$ such that $\mathrm{Eval}_{G,H} \circ (\mathrm{Inc}_{\mathcal{W}} \times I_G)$ is a homomorphism $(\mathcal{W} \times \mathcal{V}_G, I_{\mathcal{W}} \times E_G) \to H$.
2. $E_{[G,H]}$ is the largest symmetric relation R on $\mathcal{V}_{[G,H]} = \mathcal{W}$ such that $\mathrm{Eval}_{G,H} \circ (\mathrm{Inc}_{\mathcal{W}} \times I_G)$ is a homomorphism $(\mathcal{W} \times \mathcal{V}_G, R \times I_G) \to H$.

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Observations

- Proof for $(\mathbf{Gph}, \Box, K_1)$ exactly the same.
- We have a bijection between:
  - n-dimensional quantum homomorphisms $G \to H$;
  - n-dimensional atoms of $[G, H]$;

Winning quantum strategies

Theorem (Kornell, L.)

For finite simple graphs G and H the following are equivalent:

- *There is a winning quantum strategy for the (G, H)-homomorphism game;*
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Corollary

The following are equivalent for graphs G and H:

- There is a quantum homomorphism $G \to H$, but no homomorphism $G \to H$.
- $[G, H] \neq \emptyset$, but there are no one-dimensional atoms.

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