C*-algebras*

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*Based on the lectures by Tristan Bice.

1 Introduction

C*-algebra is a set together with operations: sum, product, scalar product, involution, norm. Simple examples include the complex numbers $\mathbb{C}$, matrix algebra $M_n(\mathbb{C})$, algebra of continuous functions $C([0,1],\mathbb{C})$. So, C*-algebras can be non-commutative or infinite dimensional.

C*-algebras provide an abstract way of investigating seemingly different objects like matrices and continuous functions. These two examples give us two different viewpoints:

$$\text{C*-algebras} \approx \text{infinite dimensional matrices},$$
$$\text{C*-algebras} \approx \text{non-commutative topology}.$$

By adding extra structure, one can arrive to other objects, e.g., quantum groups, non-commutative geometry, von Neumann algebras, etc.

2 Semigroups, monoids, groups

2.1 Semigroups

Definition 2.1. A *semigroup* $(S,\cdot)$ is a set S together with a binary operation $\cdot$ on S, which is *associative*, i.e., for all $a,b,c \in S$,

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c.$$

We will also write just $abc = a(bc) = (ab)c$.

Semigroups are the most basic kind of algebraic structure. They are the building blocks of other structures like C*-algebras.

Example 2.2.
- $(\mathbb{C},+)$ and $(\mathbb{C},\cdot)$ are semigroups.
- $(\mathbb{C},\odot)$, where $a\odot b = a^2b^2$ is not a semigroup.
- $(X^X,\circ)$ is a semigroup. Here, $X^X = \{f\colon X\to X\}$ and $\circ$ is the composition of functions $(f\circ g)(x) = f(g(x))$.

Definition 2.3. A semigroup S is *commutative* or *abelian* if, for all $a,b\in S$,

$$ab = ba.$$

Example 2.4.
- $(\mathbb{C},+)$ and $(\mathbb{C},\cdot)$ are commutative.
- $(M_n(\mathbb{C}),+)$ is commutative, but $(M_n(\mathbb{C}),\cdot)$ is not.
- $(X^X,\circ)$ is not commutative if $|X|\geq 2$.

Definition 2.5. An *action* of a semigroup S on a set X is a product $\cdot\colon S\times X\to X$ such that

$$s\cdot(t\cdot x) = (st)\cdot x.$$

Example 2.6.
- If S is a semigroup, then S acts on $X = S\times S$ by

$$s\cdot(a,b) = (sa,sb).$$

- $(X^X, \circ)$ acts on X by
$$f \cdot x = f(x).$$

Definition 2.7. A *subsemigroup* of semigroup S is a subset $P \subseteq S$ such that $PP \subseteq P$, i.e.,
$$p \in P \quad \text{and} \quad q \in P \implies pq \in P.$$

In other words, P is a subsemigroup of S if and only if P is a semigroup itself under the same semigroup operation.

Example 2.8.
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are subsemigroups of $\mathbb{C}$ with respect to both $\cdot$ and $+$.
- $-\mathbb{N}$ is a subsemigroup of $\mathbb{C}$ under $+$, but not under multiplication.

Definition 2.9.
- A binary relation $\prec$ on a set S is
  - *transitive* if $a \prec b \prec c \implies a \prec c$,
  - *reflexive* if $a \prec a$ for every $a \in S$,
  - *antisymmetric* if $a \prec b \prec a \implies a = b$.
- A transitive relation is an *order*.
- A transitive reflexive relation is a *preorder*.
- An antisymmetric preorder is a *partial order*.
- A partial order $\prec$ is *total* if all pairs $a, b \in S$ are comparable, i.e.,
$$a \prec b \quad \text{or} \quad b \prec a.$$

Example 2.10.
- $\subseteq$ is a partial order on all sets.
- $\leq$ is a total order on $\mathbb{R}$.

Given a semigroup S and a subsemigroup $P \subseteq S$, define
$$a \prec b \iff a \in Pb.$$

Lemma 2.11. *The relation $\prec$ on S is transitive.*

Proof. If $a \prec b \prec c$, then we have $p, q \in P$ with $a = pb$ and $b = qc$. So,
$$a = pb = p(qc) = (pq)c.$$
As P is a subsemigroup, $pq \in P$. Thus, $a \in Pc$ and hence $a \prec c$. $\square$

Example 2.12.
- $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$ form an additive subsemigroup of $\mathbb{R}$. The associated $\prec$ is
$$a \prec b \iff a \in \mathbb{R}_+ + b \iff a - b \in \mathbb{R}_+ \iff a - b \geq 0 \iff a \geq b.$$
So $\prec \, = \, \geq$ is the opposite of the usual order on $\mathbb{R}_+$.
- Consider the multiplication on $S = P = (0, 1)$. Then
$$a \prec b \iff a \in Pb \iff \frac{a}{b} \in P \iff \frac{a}{b} < 1 \iff a < b.$$
So, $\prec \, = \, <$ is the usual strict order on $(0, 1)$.

Definition 2.13. A map $\phi\colon S \to T$ between semigroups is a homomorphism if $\phi(ab) = \phi(a)\phi(b)$.

Example 2.14.
- The modulus $\phi\colon \mathbb{C} \to \mathbb{R}_+$, defined by $\phi(a) = |a|$ is a multiplicative homomorphism since $|ab| = |a||b|$.
- The modulus is not an additive homomorphism.
- The determinant $\det\colon M_n(\mathbb{C}) \to \mathbb{C}$ is a multiplicative homomorphism since $\det A \det B = \det AB$.

Definition 2.15. A *-semigroup is a semigroup together with an operation $a \mapsto a^*$ such that

$$(ab)^* = b^*a^*, \qquad \text{(antihomomorphism)}$$
$$a^{**} = a. \qquad \text{(involution)}$$

Example 2.16.
- Any commutative semigroup is a *-semigroup with $a^* = a$.
- $M_n(\mathbb{C})$ is a *-semigroup.

Definition 2.17. We call an element of a *-semigroup S

- *normal* if $aa^* = a^*a$,
- *self-adjoint* if $a = a^*$,
- *positive* if $a = bb^*$, for some $b \in S$.

Lemma 2.18. *Every positive element is self-adjoint. Every self-adjoin element is normal.*

Proof. Positive implies self-adjoint:

$$a = bb^* \implies a^* = (bb^*)^* = (b^*)^*b^* = bb^* = a.$$

Self-adjoint implies normal:

$$a = a^* \implies aa^* = aa = a^*a.$$

□

Lemma 2.19. *An element a is positive if and only if $a = b^*b$, for some $b \in S$.*

Proof. If $a = bb^* = b^{**}b^*$, then $a = c^*c$ for $c = b^*$. □

Example 2.20. Consider $(\mathbb{C}, \cdot)$ with $a^* = \overline{a}$ (complex conjugation).

- Since multiplication is commutative, every element in $\mathbb{C}$ is normal.
- The self-adjoint elements in $\mathbb{C}$ are exactly $\mathbb{R}$.
- The positive elements in $\mathbb{C}$ are exactly $\mathbb{R}_+$.

If we consider $(\mathbb{C}, +)$ instead, the normal and self-adjoint elements are the same as for $(\mathbb{C}, \cdot)$. However, the positive elements are $\mathbb{R}$.

Definition 2.21. Let S be a semigroup.

- We call an element $0 \in S$ *absorbing* if, for all $a \in S$, $0a = 0 = a0$.
- We call an element $1 \in S$ *identity* if, for all $a \in S$, $1a = a = a1$.

Note that absorbing elements and identities are unique.

2.2 Monoids

Definition 2.22. A *monoid* is a semigroup M with an identity $1 \in M$. We call $b \in M$ an *inverse* of $a \in M$ if

$$ba = 1 = ab.$$

In this case we say a is *invertible* and often write $b = a^{-1}$. Note that inverses are unique and $(a^{-1})^{-1} = a$. The invertible elements form a submonoid of M.

Example 2.23.
- $(\mathbb{C}, +, 0)$ is a monoid.
- $(\mathbb{C} \setminus \{0\}, \cdot, 1)$ is a monoid.
- $(M_n(\mathbb{C}), \cdot, I)$, where I is the identity matrix, is a monoid.

Definition 2.24. A *monoid action* is a semigroup action $\cdot : M \times X \to X$ such that for all $x \in X$,

$$1 \cdot x = x.$$

Note that if $a \in M$ has an inverse $a^{-1} \in M$, then $ax = y \implies a^{-1}y = x$.

2.3 Groups

Definition 2.25. A *group* is a monoid G where every $g \in G$ has an inverse $g^{-1} \in G$.

Example 2.26. Since $(gh)^{-1} = h^{-1}g^{-1}$ and $(g^{-1})^{-1} = g$, every G is a *-semigroup with $g^* = g^{-1}$. We determine the normal, self-adjoint and positive elemnets in G

- Normal elements are the whole G since $g^{-1}g = 1 = gg^{-1}$.
- Self-adjoin elements are elemnets of order 2 since $g^{-1} = g \iff g^2 = 1$.
- Positive is only the identity element since $1 = gg^{-1}$.

Example 2.27. $\mathrm{GL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det(A) \neq 0\}$ is a group and $\mathrm{SL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det(A) = 1\}$ is its subgroup.

Definition 2.28. We call $H \subseteq G$ a *subgroup* of a group G if

$$1 \in H, \quad HH \subseteq H, \quad \text{and} \quad H^{-1} \subseteq H.$$

Proposition 2.29. *A subset $H \subseteq G$ of a group G is a subgroup if and only if $HH^{-1} \subseteq H \neq \emptyset$.*

Proof. If H is a subgroup, then $HH^{-1} \subseteq HH \subseteq H$ and $1 \in H$, so $S \neq \emptyset$.

Conversly, assume $HH^{-1} \subseteq S \neq \emptyset$. Taking any $h \in H$, we have $1 = hh^{-1} \in HH^{-1} \subseteq H$. Thus, $H^{-1} = 1H^{-1} \subseteq HH^{-1} \subseteq H$. So, $H = (H^{-1})^{-1} \subseteq H^{-1}$ and hence $HH \subseteq HH^{-1} \subseteq H$. □

Theorem 2.30. *If G is a group then $H \subseteq G$ is a subgroup if and only if $H(G \setminus H) \subseteq G \setminus H \neq G$.*

Proof. Suppose that H is a subgroup, $h \in H$ and $g \in G \setminus H$. Then $hg \in H$ would imply $g \in h^{-1}H \subseteq H^{-1}H \subseteq HH \subseteq H$. This contradicts $g \in G \setminus H$. So $hg \notin H$ and hence $H(G \setminus H) \subseteq G \setminus H$.

Conversely, assume that $H(G \setminus H) \subseteq G \setminus H \neq G$. If $1 \notin H$, then taking any $h \in H$, we have $h = h1 \in H(G \setminus H) \subseteq G \setminus H$, contradicting $h \in H$. Thus $1 \in H$.

If $h \in H$ and $h^{-1} \notin H$, then $1 = hh^{-1} \in H(G \setminus H) \subseteq G \setminus H$. This cotradicts $1 \in H$. So, $H^{-1} \subseteq H$.

Now, if $h, k \in H$ and $hk \notin H$, then $k = h^{-1}hk \in H^{-1}(G \setminus H) \subseteq H(G \setminus H) \subseteq G \setminus H$, contradicting $k \in H$. Thus, $HH \subseteq H$ and hence H is indeed a subgroup of G. □

3 Vector spaces, algebras

3.1 Vector spaces

Definition 3.1. A *(complex) vector space* is a commutative group $(V, +, 0)$ together with a scalar product $(\alpha, v) \mapsto \alpha v \colon \mathbb{C} \times V \to V$ such that

$$\begin{aligned} \alpha(v+w) &= \alpha v + \alpha w, \\ (\alpha+\beta)v &= \alpha v + \beta v, \\ \alpha(\beta v) &= (\alpha\beta)v, \\ 1v &= v. \end{aligned}$$

In other words, $v \mapsto \alpha v$ is a homomorphism $(V, +) \to (V, +)$, for each $\alpha \in \mathbb{C}$, $\alpha \mapsto \alpha v$ is a homomorphism $(\mathbb{C}, +) \to (V, +)$, for each $v \in V$, and $(\alpha, v) \mapsto \alpha v$ is a monoid action of $(\mathbb{C}, \cdot, 1)$ on V.

Example 3.2. $\mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, $M_n(\mathbb{C})$, $C(X, \mathbb{C}) = \{\text{continuous } X \to \mathbb{C}\}$.

Definition 3.3. We call $W \subseteq V$ a *subspace* of a vector space V if

$$W \neq \emptyset, \quad \mathbb{C}W \subseteq W, \quad \text{and} \quad W + W \subseteq W.$$

Subspace is also a subgroup since

$$\{\mathbf{0}\} = 0W \subseteq \mathbb{C}W \subseteq W \quad \text{and} \quad -W = (-1)W \subseteq \mathbb{C}W \subseteq W.$$

Lemma 3.4. *W is a subspace if and only if $\emptyset \neq W = W + W = \mathbb{C}W$.*

Proof. $W = W + W = \mathbb{C}W$ implies $W + W \subseteq W$ and $\mathbb{C}W \subseteq W$. Conversely, assume that $W + W \subseteq W$ and $\mathbb{C}W \subseteq W$ hold. Then in fact $=$ holds since $W = 1W \subseteq \mathbb{C}W$ and $W = \mathbf{0} + W \subseteq W + W$. □

Lemma 3.5. *If $\mathcal{F}$ is a family of subspaces, then $\bigcap \mathcal{F}$ is also a subspace.*

Proof. Take $v, w \in \bigcap \mathcal{F}$. This means that $v, w \in W$, for every $W \in \mathcal{F}$. As W is a subspace, $v + w \in W$. As W was arbitrary, $v + w \in \bigcap \mathcal{F}$. Thus, $\bigcap \mathcal{F} + \bigcap \mathcal{F} \subseteq \bigcap \mathcal{F}$. Likewise, $\mathbb{C} \bigcap \mathcal{F} \subseteq \bigcap \mathcal{F}$. □

Definition 3.6. The *span* of any $S \subseteq V$ is the smallest subspace containing S, i.e.,

$$\operatorname{span}(S) = \bigcap \{W : S \subseteq W \text{ and } W \text{ is a subspace of } V\}.$$

Equivalently, $\operatorname{span}(S)$ is *generated* by S, i.e.,

$$\operatorname{span}(S) = \left\{ \sum_{k=1}^{n} \alpha_k s_k : \alpha_1, \ldots, \alpha_n \in \mathbb{C} \text{ and } s_1, \ldots, s_n \in S \right\}.$$

Note that the *exception* is $S = \emptyset$, in which case $\operatorname{span}(\emptyset) = \{0\}$.

Definition 3.7. The *dimension* of V is the smallest cardinality of a spanning subset, i.e.,

$$\dim(V) = \min\{|S| : \operatorname{span}(S) = V\}.$$

Example 3.8.
- $\dim(\{0\}) = 0$, $\dim(\mathbb{C}) = 1$, $\dim(\mathbb{C} \times \mathbb{C}) = 2$, $\dim(M_2(\mathbb{C})) = 4$.
- Some vector spaces have infinite dimesnion, e.g., the space $c_{00}(\mathbb{C})$ of sequences of complex numbers that are eventually zero,

$$c_{00}(\mathbb{C}) = \{(\alpha_1, \alpha_2, \dots) : \exists n \in \mathbb{N}\ \forall k \geq n\ (\alpha_k = 0)\}.$$

  Note that $c_{00}(\mathbb{C}) = \bigcup_{n \in \mathbb{N}} V_n$, where $V_n \approx \mathbb{C}^n$ is the subspace

$$V_n = \{(\alpha_1, \alpha_2, \dots) : \forall k \geq n\ (\alpha_k = 0)\}.$$

  Since (V_n) is strictly incerasing, with respect to $\subseteq$, $c_{00}(\mathbb{C})$ has infinite dimension.

Definition 3.9. We call $S \subseteq V$ *linearly independent* if S has no proper spanning subset, i.e.,

$$R \subsetneq S \implies \mathrm{span}(R) \subsetneq \mathrm{span}(S).$$

Equivalently, $S \subseteq V$ is linearly independent if and only if whenever we have finite distinct $s_1, \dots, s_n \in S$ and finite $\alpha_1, \dots, \alpha_n \in \mathbb{C}$,

$$\alpha_1 s_1 + \cdots \alpha_n s_n = 0 \implies \alpha_1 = \cdots = \alpha_n = 0.$$

Definition 3.10. A *basis* $B \subseteq V$ is linearly independent spanning set, i.e. $V = \mathrm{span}(B)$.

Theorem 3.11 (Kuratowski-Zorn Lemma). *If $(\mathbb{P}, \leq)$ is a partially ordered set such that every totally ordered subset $\mathbb{T} \subseteq \mathbb{P}$ has an upper bound $p \in \mathbb{P}$, then $\mathbb{P}$ has a maximal element.*

Proposition 3.12. *Every vector space has a basis.*

Proof. Let $\mathbb{P} = \{S \subseteq V : S \text{ is linearly independent}\}$. Any totally ordered $\mathbb{T} \subseteq \mathbb{P}$ has an upper bound. Specifically, $S = \bigcup \mathbb{T}$ is linearly independent. If not, then we would have $s \in S$ with $s \in \mathrm{span}(S \setminus \{s\})$. But then $s \in \mathrm{span}(F)$, for some finite $F \subseteq S \setminus \{s\}$. For each $f \in F \cup \{s\}$, we have some $T \in \mathbb{T}$ with $f \in T$. Since $\mathbb{T}$ is totally ordered and F is finite, these T's have a maximum, i.e., we have some $T \in \mathbb{T}$ with $F \cup \{s\} \subseteq T$. But then $s \in \mathrm{span}(F) \subseteq \mathrm{span}(T \setminus \{s\})$, contradicting $T \in \mathbb{T}$.

By Theorem 3.11, there is a maximal linearly independent subset $S \subseteq V$. By maximality, if $v \in V \setminus S$, then $S \cup \{v\}$ is not linearly independent, so $v \in \mathrm{span}(S)$. Since v was arbitrary, $V \setminus S \subseteq \mathrm{span}(S)$. Certainly, $S \subseteq \mathrm{span}(S)$, so, $V = S \cup (V \setminus S) = \mathrm{span}(S)$. $\square$

Proposition 3.13. *If $B \subseteq V$ is a basis of a vector space V, then $\dim(V) = |B|$.*

Example 3.14.
- $\dim(c_{00}(\mathbb{C})) = |\mathbb{N}|$, $c_{00}(\mathbb{C})$ has standard basis $(e_n)_{n \in \mathbb{N}}$, where $e_n = (0, \dots, 0, 1, 0, \dots)$ with 1 on the nth position.
- $(e_n)_{n \in \mathbb{N}}$ is not a basis of $c_0(\mathbb{C}) = \{(\alpha_1, \alpha_2, \dots) : \alpha_n \to 0\}$ since $(1, 1/2, 1/3, \dots) \notin \mathrm{span}(e_n)_{n \in \mathbb{N}}$. In fact, $\dim(c_0(\mathbb{C})) = |\mathbb{R}| > |\mathbb{N}|$.

Definition 3.15. Let V and W be vector spaces. We call $T \colon V \to W$ *linear* if T is a vector space homomorphism, i.e.,

$$T(v + w) = T(v) + T(w),$$
$$T(\alpha v) = \alpha T(v).$$

Let $\mathcal{L}(V) = \{T \in V^V : T \text{ is linear}\}$, where $\dim(V) = n$. If we fix a basis of V, then there is a matrix representation M_T of T. So $T \mapsto M_T$ is a homomorphism from $(\mathcal{L}(V), \circ) \to (M_n(\mathbb{C}), \cdot)$. Since we can also define $T_M \in \mathcal{L}(V)$, for any $M \in M_n(\mathbb{C})$, $T \mapsto M_T$ is an *isomorphism*. Changing the fixed basis, changes the isomorphism.

Definition 3.16. An *inner product* is a map $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ such that

$$\begin{aligned} \langle v + w, x \rangle &= \langle v, x \rangle + \langle w, x \rangle, \\ \langle \alpha v, w \rangle &= \alpha \langle v, x \rangle, \\ \langle v, w \rangle &= \overline{\langle w, v \rangle}, \\ v \neq 0 &\Rightarrow \langle v, v \rangle > 0. \end{aligned}$$

The map $v \mapsto \langle v, x \rangle$ is linear, for $x \in V$, and $\langle \cdot, \cdot \rangle$ is *conjugate-symmetric*. The map $v \mapsto \langle x, v \rangle$ is *conjugate-linear*.

Example 3.17. The standard inner product on $\mathbb{C}^n$ is $\langle \cdot, \cdot \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w_i}.$$

If $M \in M_n(\mathbb{C})$, then

$$\langle v, Mw \rangle = \langle M^* v, w \rangle,$$

where M^* is the conjugate transpose of M. This is the * operation on matrices. Generally, if $\dim(V) < \infty$ and $\langle \cdot, \cdot \rangle$ is an inner product on V, then every $T \in \mathcal{L}(V)$ has unique *adjoint*, i.e., $T^* \in \mathcal{L}(V)$ such that

$$\langle v, Tw \rangle = \langle T^* v, w \rangle.$$

Definition 3.18. A **-vector space* is a vector space V together with a map $^* \colon V \to V$ such that

$$\begin{aligned} (v + w)^* &= v^* + w^*, \\ (\alpha v)^* &= \overline{\alpha} v^*. \end{aligned}$$

In other words, $v \mapsto v^*$ is conjugate-linear. Note that $(V, +, ^*)$ is a *-semigroup since $(v+w)^* = v^* + w^* = w^* + v^*$.

Example 3.19. $\mathbb{C}^3$ with $(\alpha_1, \alpha_2, \alpha_3)^* = (\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3})$ is a *-vector space.

3.2 Algebras

Definition 3.20. A **-algebra* is a *-vector space $(A, +, ^*)$ and a *-semigroup $(A, \cdot, ^*)$ such that

$$(a + b)c = ac + bc \quad \text{and} \quad (\alpha b)c = \alpha(bc).$$

In other words, $a \mapsto ac$ is linear, for each $c \in A$.

Lemma 3.21. *The map $a \mapsto ca$ is linear.*

Proof.

$$\begin{aligned} c(a+b) &= (c(a+b))^{**} = ((a+b)^*c^*)^* = ((a^*+b^*)c^*)^* \\ &= (a^*c^* + b^*c^*)^* = ((ca)^* + (cb)^*)^* = ca + cb. \\ c(\alpha a) &= (c(\alpha a))^{**} = ((\alpha a)^*c^*)^* = (\overline{\alpha} a^* c^*)^* = (\overline{\alpha}(ca)^*)^* = \alpha ca. \end{aligned}$$

□

Example 3.22.
- $M_n(\mathbb{C})$, or equivalently $\mathcal{L}(V)$, for $\dim(V) < \infty$, is a *-algebra.
- $C(X, \mathbb{C})$, $C(X, M_n(\mathbb{C}))$ are *-algebras.

Definition 3.23. A *metric* on a set X is a function $d\colon X^2 \to [0, \infty)$ such that

$$\begin{aligned} & d(x,y) = d(y,x), && \text{(symmetry)} \\ & d(x,y) = 0 \iff x = y, && \text{(coincidence)} \\ & d(x,y) \le d(x,z) + d(z,y). && \text{(triangle inequality)} \end{aligned}$$

Definition 3.24. A metric d on a vector space V is *compatible* if

$$\begin{aligned} d(v+x, w+x) &= d(v,w), && \text{(translation invariant)} \\ d(\alpha v, \alpha w) &= |\alpha| d(v,w). && \text{(homogeneous)} \end{aligned}$$

Definition 3.25. A function $\|\cdot\|\colon V \to [0, \infty)$ is a *norm* if

$$\begin{aligned} \|v\| = 0 &\implies v = 0, && \text{(definite)} \\ \|v + w\| &\le \|v\| + \|w\|, && \text{(subadditive)} \\ \|\alpha v\| &= |\alpha| \|v\|. && \text{(homogeneous)} \end{aligned}$$

Lemma 3.26. *There is a norm $\|\cdot\|$ on V if and only if there is a compatible metric d on V.*

Proof. Let d be a compatible metric on V. Define $\|v\| = d(v, 0)$. The translation invariance gives

$$d(v,w) = d(v + (-w), w + (-w)) = d(v - w, 0) = \|v - w\|.$$

Further,

$$\begin{aligned} 0 = \|v\| = d(v, 0) &\iff v = 0, \\ \|\alpha v\| = d(0, \alpha v) &= d(\alpha 0, \alpha v) = |\alpha| d(0, v) = |\alpha| \|v\|, \\ \|v + w\| = d(0, v+w) &\le d(0,v) + d(v, v+w) = d(0,v) + d(0,w) = \|v\| + \|w\|. \end{aligned}$$

Thus, $\|v\| = d(v, 0)$ is a norm.

Conversely, if $\|\cdot\|$ is a norm, then $d(v,w) = \|v - w\|$ is compatible.

$$\begin{aligned} 0 = d(v,w) = \|v - w\| &\iff v - w = 0 \iff v = w, \\ d(v,w) = \|v - w\| &= |-1| \|v - w\| = \|w - v\| = d(w, v), \\ d(v,w) = \|v - w\| &= \|v - x + x - w\| \le \|v - x\| + \|x + w\| = d(v,x) + d(x,w). \end{aligned}$$

Since $d(v, 0) = \|v - 0\| = \|v\|$, these constructions are mutually inverse. □

Definition 3.27. A *normed space* is a vector space together with a norm $\|\cdot\|$.

Definition 3.28. A *normed *-algebra* is a *-algebra A together with a norm $\|\cdot\|$ satisfying

$$\begin{aligned} &\|ab\| \le \|a\|\|b\|, && \text{(submultiplicative)} \\ &\|a^*\| = \|a\|. && \text{(*-invariant)} \end{aligned}$$

3.3 C*-algebras

Definition 3.29. We call a sequence $(v_n)_{n\in\mathbb{N}} \subseteq V$ in a metric space V *Cauchy* if

$$\forall \varepsilon > 0\ \exists k \in \mathbb{N}\ \forall j \ge k\ (d(v_j, v_k) < \varepsilon),$$

or, equivalently,

$$\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \forall j, k \ge M\ (d(v_j, v_k) < \varepsilon).$$

V is *complete* if every Cauchy $(v_n) \subseteq V$ converges to some $w \in V$, i.e.,

$$\forall \varepsilon > 0\ \exists k \in \mathbb{N}\ \forall j \ge k\ (d(v_j, w) < \varepsilon).$$

Example 3.30. $\mathbb{R}$ and $\mathbb{C}$ are complete with respect to $d(v, w) = |v - w|$.

Definition 3.31. A *Banach space* is a complete normed space. A *Banach *-algebra* is a complete normed *-algebra. A *C*-algebra* is a Banach *-algebra such that

$$\|a^*a\| = \|a^*\|\|a\|. \qquad \text{(*-multiplicative)}$$

Observation 3.32. **-invariant and *-multiplicative hold if and only if*

$$\|a^*a\| = \|a\|^2. \qquad (C^*)$$

Example 3.33. For any set X and $f \in \mathbb{C}^X$, define $\|f\| = \sup_{x\in X} |f(x)|$ and $f^* = \overline{f}$. Then

$$\ell^\infty(X) = \left\{f \in \mathbb{C}^X : \|f\| < \infty\right\}$$

is a commutative C*-algebra. Each $f \in \ell^\infty(X)$ can be veiwed as a linear operator on

$$\ell^2(X) = \left\{g \in \mathbb{C}^X : \langle g, g\rangle < \infty\right\},$$

the inner product spece where $\langle g, h\rangle = \sum_{x\in X} g(x)\overline{h(x)}$, namely

$$T_f g = fg.$$

Moreover, $\langle fg, h\rangle = \left\langle g, \overline{f}h\right\rangle = \langle g, f^*h\rangle$ and $\|f\| = \sup_{\|g\|_2=1} \|fg\|_2$. We need to consider more general linear operators to obtain non-commutative C*-algebras.

Example 3.34. If X is a metric space, then

$$C^b(X) = \{f \in \ell^\infty(X) : f \text{ is continuous}\}$$

is a C*-subalgebra of $\ell^\infty(X)$.

Proposition 3.35. *In a normed space, $f(x) = \|x\|$ defines a continuous function.*

Proof. $\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$. So, $\|x\| - \|y\| \le \|x - y\|$. Likewise, $\|y\| - \|x\| \le \|y - x\| = \|x - y\|$. So, $|\|x\| - \|y\|| \le \|x - y\|$. Taking $y = x_n$, we see that $\|x - x_n\| \to 0$ implies $|\|x\| - \|x_n\|| \to 0$. ☐

Definition 3.36. For any linear map $T \in W^V$ between normed spaces, we define the *operator norm* of T as

$$\|T\| = \sup_{\|x\| \le 1} \|T(x)\| = \sup_{x \ne 0} \frac{\|T(x)\|}{\|x\|}.$$

This defines a norm.

Proposition 3.37. *T is continuous if and only if T is bounded, i.e., $\|T\| < \infty$.*

Proof. If T is bounded and $x_n \to x$, then $T(x_n) \to T(x)$ since

$$\|T(x_n) - T(x)\| = \|T(x_n - x)\| \le \|T\|\|x_n - x\| \to 0.$$

Conversely, if T is not bounded, we have $(x_n) \subseteq V^1$ with $\|T(x_n)\| \to \infty$. Then

$$y_n = \frac{1}{\|T(x_n)\|} x_n \to 0$$

while

$$T(y_n) = \left\| T\left(\frac{1}{\|T(x_n)\|} x_n\right)\right\| = \frac{\|T(x_n)\|}{\|T(x_n)\|} = 1.$$

Thus $T(y_n) \not\to 0 = T(0)$ and hence T is not continuous. ☐

We denote the bounded linear operators on a normed space V by

$$\mathcal{B}(V) = \{T \in \mathcal{L} : \|T\| < \infty\} = \left\{T \in V^V : T \text{ is linear and } \|T\| < \infty\right\}.$$

Theorem 3.38. *If V is a Banach space the $\mathcal{B}(V)$ is a Banach algebra.*

Proof. $\mathcal{B}(V)$ is a normed algebra, we need to show completeness. Let $(T_n) \subseteq \mathcal{B}(V)$ be Cauchy, i.e., $\|T_m - T_n\| \to 0$ as $m, n \to \infty$. Then for each $v \in V$, we have $\|T_m(v) - T_n(v)\| \le \|T_m - T_n\|\|v\| \to 0$. So, the sequence $(T_n(v))$ is Cauchy. Since V is complete, we can define $T(v) = \lim T_n(v)$. Linearity of T follows from the linearity of each T_n. By Proposition 3.35, norm is continuous, and $\|Tv\| \le \sup_n \|T_n(v)\| \le \sup_n \|T_n\|\|v\|$. Thus $\|T\| \le \sup_n \|T_n\| < \infty$ since (T_n) is Cauchy. So, $T \in \mathcal{B}(V)$. ☐

Definition 3.39. A *sesquilinear form* on a vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that is linear in the first coordinate and conjugate-linear in the second, i.e.,

$$\begin{aligned} \langle v + w, x\rangle &= \langle v, x\rangle + \langle w, x\rangle, \\ \langle v, x + y\rangle &= \langle v, x\rangle + \langle v, y\rangle, \\ \langle \lambda v, x\rangle &= \lambda\langle v, x\rangle = \langle v, \overline{\lambda} x\rangle. \end{aligned}$$

A sesquilinear form is *real/positive/strictly positive* if for all $x \in V \setminus \{0\}$, $\langle x, x\rangle \in \mathbb{R}/\mathbb{R}_+/\mathbb{R}_+\setminus\{0\}$. Every sesquilinear form is *conjugate-symmetric*: $\langle x, y\rangle = \overline{\langle y, x\rangle}$. Inner product is strictly positive sesquiliner form. For a positive sesquilinear form, we define $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 3.40 (Caychy-Schwartz Inequality). *For any positive sesquilinear form,*

$$|\langle v, w\rangle| \le \|v\|\|w\|.$$

Proof. Since $\langle \cdot, \cdot \rangle$ is positive, $\langle z, z\rangle \ge 0$ for $z = \langle w, w\rangle v - \langle v, w\rangle w$. It sufficies to exapnd this using other inner product properties. $\square$

Corollary 3.41. *In any inner product space, $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.*

Proof. $\|\cdot\|$ is homogeneous since $\langle \cdot, \cdot \rangle$ is sesquilinear. $\|\cdot\|$ is definite since $\langle \cdot, \cdot \rangle$ is positive definite. $\|\cdot\|$ is subadditive by Theorem 3.40. $\square$

Lemma 3.42. *Inner product spaces are exactly normed spaces satisfying*

$$2\|v\|^2 + 2\|w\|^2 = \|v + w\|^2 + \|v - w\|^2. \qquad \textit{(parallelogram law)}$$

Definition 3.43. A *Hilbert space* is a complete inner product space, i.e., complete with respect to

$$d(v, w) = \|v - w\| = \sqrt{\langle v - w, v - w\rangle}.$$

Example 3.44. $\ell^2(X) = \left\{g \in \mathbb{C}^X : \langle g, g\rangle < \infty\right\}$ is a Hilbert space where $\langle g, h\rangle = \sum_{x\in X} g(x)\overline{h(x)}$. By Theorem 3.40, this is well defined inner product:

$$\sum_{x\in X} |g(x)\overline{h(x)}| = \sum_{x\in X} |g(x)||h(x)| \le \sqrt{\sum_{x\in X} |g(x)|^2}\sqrt{\sum_{x\in X} |h(x)|^2} < \infty.$$

Definition 3.45. We call a subset C of a vector space V *convex* if

$$\lambda \in [0, 1] \quad \text{and} \quad x, y \in C \implies \lambda x + (1 - \lambda)y \in C.$$

We call a subset C of a metric space V *closed* if

$$C \supseteq x_n \to y \implies y \in C.$$

Theorem 3.46. *If H is a Hilbert space, then for any $v \in H$ and closed convex $C \subsetneq H$, we can find a point $w \in C$ minimizing the distance to v, i.e.,*

$$\|v - w\| = \inf_{x\in C} \|v - x\|.$$

Proof. Take a sequence $(x_n) \subseteq C$ with $\|v - x_n\| \to d = \inf_{x\in C} \|v - x\|$. So for $\varepsilon > 0$, we have $m \in \mathbb{N}$ such that $\|v - x_n\| \le d + \varepsilon$, for $n \ge m$. Then

$$\begin{aligned}
\|x_n - x_m\|^2 &= \|x_n - v + v - x_n\|^2 \\
&= 2\|x_n - v\|^2 + 2\|v - x_m\|^2 - \|(x_n - v) - (v - x_m)\|^2 \qquad \text{(parallelogram law)} \\
&= 2\|v - x_n\|^2 + 2\|v - x_m\|^2 - 4\left\|\frac{1}{2}(x_n + x_m) - v\right\|^2 \\
&\le 2(d + \varepsilon) + 2(d + \varepsilon) - 4d = 4\varepsilon.
\end{aligned}$$

Thus (x_n) is Cauchy and hence $x_n \to w$, for some $w \in H$. Since C is closed, we have $w \in C$. $\square$

Definition 3.47. For a Hilbert space H and $S \subseteq H$, let

$$S^{\perp} = \{y \in H : \forall s \in S \ (\langle s, y\rangle = 0)\}$$

be the *orthogonal complement* of S in H.

Corollary 3.48. *If V is a closed subspace of a Hilbert space H, then $H = V + V^{\perp}$.*

Proof. By Theorem 3.46, for any $x \in H$, there is $w \in V$ with $\|x - w\| = \inf_{v \in V} \|x - v\|$. Thus, for any $v \in V$, we have

$$\begin{aligned}
\langle x - w, x - w\rangle &\leq \langle x - (w + v), x - (w + v)\rangle, \\
\langle y, y\rangle &\leq \langle y - v, y - v\rangle, && \text{(substituting } y = x - w\text{)} \\
\langle y, y\rangle &\leq \langle y, y\rangle - \langle y, v\rangle - \langle v, y\rangle + \langle v, v\rangle. \\
2\Re(\langle y, v\rangle) &\leq \langle v, v\rangle.
\end{aligned}$$

Replacing v with tv for $t \in \mathbb{R}$ or $t \in i\mathbb{R}$ shows $\langle y, v\rangle = 0$. Thus, $x = x - w + w = y + w \in V^{\perp} + V$. □

Theorem 3.49 (Riesz Representation). *If H is a Hilbert space, then every continuous linear $\phi \in \mathbb{C}^H$ is defined by*

$$\phi(x) = \langle x, v\rangle, \quad \textit{for a unique } v \in H..$$

Proof. Since ϕ is continuous, $V = \{x \in H : \phi(x) = 0\}$ is closed. Thus $H = V + V^{\perp}$. Note that $V^{\perp}$ has dimension at most 1. If $V^{\perp} = \{0\}$, then $V = H$ and we can take $v = 0$. Otherwise, take $w \in V^{\perp}$ with $\|w\| = 1$ and let $v = \overline{\phi(w)}w$. Then

$$\langle w, v\rangle = \left\langle w, \overline{\phi(w)}w\right\rangle = \phi(w)\langle w, w\rangle = \phi(w).$$

Also, for any $x \in V$,

$$\langle x, v\rangle = \phi(w)\langle x, w\rangle = 0 = \phi(x).$$

Thus, ϕ and $\langle \cdot, v\rangle$ agree on V and $V^{\perp}$ and hence on H.

For uniqueness, suppose that $\langle x, v\rangle = \langle x, w\rangle$, for all $x \in H$. Then $\langle x, v - w\rangle = 0$, so $\langle v - w, v - w\rangle = 0$, and hence $v - w = 0$. □

Theorem 3.50. *For every bounded linear operator T on a Hilbert space H, there is another* adjoint *bounded linear operator T^* on H satisfying*

$$\langle Tv, w\rangle = \langle v, T^* w\rangle.$$

Proof. From Theorem 3.40 and boundedness of T, we get $|\langle Tv, w\rangle| \leq \|T\|\|v\|\|w\|$. So $\phi_w(v) = \langle Tv, w\rangle$ is bounded. By Theorem 3.49, we have unique $u \in H$ such that $\langle Tv, w\rangle = \phi_w(v) = \langle v, u\rangle$. The define the mapping T^* by $T^*(w) = u$. The mapping T^* defined in such a way is linear and $\|T^*\| \leq \|T\|$ since

$$\|T^* w\| = \sup_{v \in H^1} |\langle v, T^* w\rangle| = \sup_{v \in H^1} |\langle Tv, w\rangle| \leq \|T\|\|w\|.$$

□

It is easy to check that $T^{**} = T$ and $\|T^* T\| = \|T\|^2$. We get the following

Corollary 3.51. *$\mathcal{B}(H) = \{T \in \mathcal{L}(H) : \|T\| < \infty\}$ is a C^*-algebra.*

4 The spectrum

4.1 Unital algebras

Definition 4.1. The *spectrum* of an element b of a unital algebra A is given by

$$\sigma(b) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - b \text{ is not invertible}\}.$$

Example 4.2.

- If $A = \mathbb{C}^X$ with pointwise operations, then for any $f \in A$,

  $$\sigma(f) = \operatorname{ran}(f) = \{f(x) : x \in X\}.$$

  To see this, let $f(x) = \lambda$. Then for all $g \in A$,

  $$((\lambda\mathbf{1} - f)g)(x) = (\lambda - f(x))g(x) = 0.$$

  Thus $(\lambda\mathbf{1} - f)g \neq \mathbf{1}$ so $\lambda\mathbf{1} - f$ is not invertible, i.e., $\lambda \in \sigma(f)$. Conversely, if $\lambda \notin \operatorname{ran}(f)$, then for all $x \in X$, we can define

  $$g(x) = \frac{1}{\lambda - f(x)}.$$

  Since $(\lambda\mathbf{1} - f)g = \mathbf{1}$, g is the inverse of $(\lambda\mathbf{1} - f)$, i.e., $\lambda \notin \sigma(f)$.

- For general algebras, we can think of $\sigma(a)$ as an "abstract range".

Definition 4.3. If V is a vector space, the *point-spectrum* of any $T \in \mathcal{L}(V)$ is given by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda\mathbf{1} - T) \neq \{0\}\}.$$

So, $\lambda \in \sigma_p(T)$ means that there is a non-zero $w \in \operatorname{Ker}(\lambda\mathbf{1} - T)$, i.e., $T(w) = \lambda w$. In other words, w is an *eigenvector* of T with *eigenvalue* λ. Then $(S(\lambda\mathbf{1} - T))(w) = 0$, for all $S \in \mathcal{L}(V)$, so

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\} \subseteq \sigma(T).$$

Observation 4.4. *$\sigma_p(T) = \sigma(T)$, for all $T \in \mathcal{L}(V)$, if and only if $\dim(V) < \infty$.*

Definition 4.5. The *spectral radius* of an element b of a unital algebra A is given by

$$|b|_\sigma = \sup\{|\lambda| : \lambda \in \sigma(b)\}.$$

Example 4.6.

- If $f \in A = \mathbb{C}^X$, then $|f|_\sigma = \sup_{x \in X} |f(x)| = \|f\|$.
- If $T \in \mathcal{L}(V)$, where $\dim(V) < \infty$, then

  $$|T|_\sigma = \max\{|\lambda| : Tw = \lambda w\}.$$

- If

  $$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

  then $|T|_\sigma = 0 < 1 = \|T\|$.

- If $Tw = \lambda w$, for $w \neq 0$, then for any $n \in \mathbb{N}$, $|\lambda| \leq \|T^n\|^{\frac{1}{n}}$ since
$$|\lambda|^n \|w\| = \|\lambda^n w\| = \|T^n w\| \leq \|T^n\| \|w\|.$$
Thus, if $\dim(V) < \infty$, then
$$|T|_\sigma \leq \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}.$$

Proposition 4.7. *For any b in a normed algebra A (or even a "normed semigroup")*
$$\|b^n\|^{\frac{1}{n}} \to \inf_{n \in \mathbb{N}} \|b^n\|^{\frac{1}{n}} \quad \text{as} \quad n \to \infty.$$

Proof. Given $m \in \mathbb{N}$, write $n = d_m m + r_m$, for $0 \leq r_m < m$. So,
$$\|b^n\|^{\frac{1}{n}} \leq \left\| (b^m)^{d_m} b^{r_m} \right\|^{\frac{1}{n}} \leq \|b^m\|^{\frac{d_m}{n}} \|b\|^{\frac{r_m}{n}} = \left(\|b^m\|^{\frac{1}{m}} \right)^{\frac{d_m m}{n}} \|b\|^{\frac{r_m}{n}}.$$
As $n \to \infty$,
$$\frac{d_m m}{n} = \frac{n - r_m}{n} < \frac{n - m}{n} \to 1 \quad \text{and} \quad \frac{r_m}{n} < \frac{m}{n} \to 0.$$
Thus,
$$\left(\|b^m\|^{\frac{1}{m}} \right)^{\frac{d_m m}{n}} \|b\|^{\frac{r_m}{n}} \to \|b^m\|^{\frac{1}{m}}.$$
So for any $\varepsilon > 0$, we eventually have
$$\|b^n\|^{\frac{1}{n}} \leq \|b^m\|^{\frac{1}{m}} + \varepsilon.$$
Since m was arbitrary, this shows that the infimum is a limit. □

4.2 General spectrum

To extend σ to non-unital algebra A, we consider $\odot$ on A given by
$$a \odot b = a + b - ab.$$

Observation 4.8. *The operation $\odot$ is associative with identity $\mathbf{0}$: $(A, \odot)$ is a monoid.*

If A is unital, then $\mathbf{1} - a \odot b = \mathbf{1} - a - b + ab = (\mathbf{1} - a)(\mathbf{1} - b)$. So,
$$a \mapsto \mathbf{1} - a$$
is a semigroup isomorphism $(A, \odot) \to (A, \cdot)$. So $\mathbf{1} - b$ is invertible in $(A, \cdot)$ if and only if b is invertible in $(A, \odot)$. When $\lambda \neq 0$, $\lambda\mathbf{1} - b = \lambda^{-1}(\mathbf{1} - \lambda^{-1} b)$, so this means
$$\sigma(b) \setminus \{0\} = \left\{ \lambda : \lambda^{-1} b \text{ is not } \odot\text{-invertible} \right\}.$$
This makes sense even for non-unital A.

Definition 4.9. The non-zero *spectrum* of an element b of any algebra A is given by
$$\sigma(b) \setminus \{0\} = \left\{ \lambda : \lambda^{-1} b \text{ is not } \odot\text{-invertible} \right\}.$$
We declare $0 \in \sigma(b)$ if and only if b is invertibles with respect to the usual product $\cdot$. In particular, if A is not unital, then $0 \in \sigma(b)$, for all $b \in A$.

Example 4.10. If $A = c_0(\mathbb{N}) = \{f \in \mathbb{C}^{\mathbb{N}} : f(n) \to 0\}$, with pointwise operations, then

$$\sigma(f) \setminus \{0\} = \mathrm{ran}(f) \setminus \{0\}.$$

To see this, let $f(x) = \lambda \neq 0$. Then for all $g \in A$,

$$(\lambda^{-1} f \odot g)(x) = \lambda^{-1} f(x) + g(x) - \lambda^{-1} f(x) g(x) = 1 + g(x) - g(x) = 1.$$

Thus, $(\lambda^{-1} f \odot g)(x) \neq \mathbf{0}$ so $\lambda^{-1} f$ is not $\odot$-invertible, i.e., $\lambda \in \sigma(f)$. Conversely if $\lambda \notin \mathrm{ran}(f)$, then $\lambda^{-1} f$ has an $\odot$-inverse g defined by

$$g(x) = \frac{f(x)}{f(x) - \lambda}.$$

We will denote the $\odot$ inverse of b by b°.

Theorem 4.11. *In any algebra A, if $s = -\sum_{n=1}^{\infty} (b^\circ \odot a)^n$ is defined, then*

$$a \odot s = b.$$

Proof. Note that

$$s = -b^\circ \odot a + (b^\circ \odot a)s$$

and

$$b = a - b^\circ \odot a + b(b^\circ \odot a).$$

So,

$$\begin{aligned}
a \odot s &= a + s - as \\
&= a + s + (b - a)s - bs \\
&= a - b^\circ \odot a + (b^\circ \odot a)s + (b - a)s + b(b^\circ \odot a) - b(b^\circ \odot a)s \\
&= a - b^\circ \odot a + b(b^\circ \odot a) + (b^\circ \odot a + b - a - b(b^\circ \odot a))s \\
&= a - b^\circ \odot a + b(b^\circ \odot a) + (b - a + b^\circ \odot a - b(b^\circ \odot a)) \\
&= b.
\end{aligned}$$

□

Corollary 4.12. *If A is a Banach algebra then, for any $b \in A$,*

$$\inf_{n \in \mathbb{N}} \|b^n\|^{\frac{1}{n}} < 1 \quad \Longrightarrow \quad b \text{ is } \odot\text{-invertible.}$$

Proof. Take γ with $\inf_{b \in \mathbb{N}} \|b^n\|^{\frac{1}{n}} < \gamma < 1$. Since "inf = lim" here, for large j, we have

$$\left\| \sum_{n=j}^{k} b^n \right\| \leq \sum_{n=j}^{k} \|b^n\| \leq \sum_{n=j}^{k} \gamma^n \leq \frac{\gamma^j}{1 - \gamma} \to 0 \quad \text{as} \quad j \to \infty.$$

So, since A is complete, $s = -\sum_{n=1}^{\infty} b^n = -\sum_{n=1}^{\infty} (0^\circ \odot b)^n$ is defined. By Theorem 4.11, $b \odot s = 0$. Likewise $s \odot b = 0$, so $s = b^\circ$. □

Corollary 4.13. *If A is a Banach algebra, then, for any $b \in A$,*

$$|b|_\sigma \leq \inf_{n \in \mathbb{N}} \|b^n\|^{\frac{1}{n}}.$$

Proof. For any $\lambda > \inf_n \|b^n\|^{\frac{1}{n}}$,

$$\inf_n \left\|(\lambda^{-1}b)^n\right\|^{\frac{1}{n}} = \inf_n \left\|\lambda^{-n}b^n\right\|^{\frac{1}{n}} = \inf_n |\lambda|^{-1}\|b^n\|^{\frac{1}{n}} < 1.$$

Thus $\lambda^{-1}b$ has a $\odot$-inverse by Corollary 4.12. So $\lambda \notin \sigma(b)$, by the definition of spectrum. $\square$

Corollary 4.14. *In any normed algebra A, $b \mapsto b^\circ$ is continuous whenever it is defined.*

Proof. By taking the completion, we may assume A is a Banach algebra. If $\|b\| < 1$, then

$$\|b^\circ\| = \left\| -\sum_{n=1}^{\infty} b^n \right\| \leq \sum_{n=1}^{\infty} \|b\|^n \leq \frac{\|b\|}{1 - \|b\|}.$$

Thus, if $\|b\| \to 0$, then also $\|b^\circ\| \to 0$, i.e., $b \mapsto b^\circ$ is continuous at $\mathbf{0}$.

Now, fix $\odot$-invertible $b \in A$. If $a \to b$, then $b^\circ \odot a \to b^\circ \odot b \to \mathbf{0}$. So,

$$s_a = -\sum_{n=1}^{\infty} (b^\circ \odot a)^n \to \mathbf{0}.$$

From $a \odot s_a = b$, we get $a^\circ = s_a \odot b^\circ \to b^\circ$, as $a \to b$. Thus, $a \mapsto a^\circ$ is also continuous at b. $\square$

Lemma 4.15. *If $b \in A$, for a Banach algebra A, then $\sigma(b)$ is compact.*

Theorem 4.16. *If A is a (complex) normed algebra, then, for any $b \in A$,*

$$|b|_\sigma \geq \inf_{n \in \mathbb{N}} \|b^n\|^{\frac{1}{n}}.$$

Proof. For contradiction, assume that $|b|_\sigma < \inf_n \|b^n\|^{\frac{1}{n}} = r$. Then the $\odot$-inverse $(\lambda^{-1}b)^\circ$ is defined whenever $|\lambda| \geq r$. So, $\lambda \mapsto (\lambda^{-1}b)^\circ$ is uniformly continuous on

$$\{\lambda \in \mathbb{C} : r \leq |\lambda| \leq r + 1\}.$$

Take any such λ.

Let $\alpha^1, \dots, \alpha^n$ be the n^{th} roots of 1 in $\mathbb{C}$. Let $\lambda_k = \alpha^k \lambda$, for $1 \leq k \leq n$. By the factor theorem,

$$1 - (\lambda^{-1}x)^n = \prod_{k=1}^{n} (1 - \lambda_k^{-1}x).$$

In the polynomial algebra $\mathbb{C}[x]$, this is equivalent to

$$(\lambda^{-1}x)^n = \lambda_1^{-1}x \odot \lambda_2^{-1}x \odot \cdots \odot \lambda_n^{-1}x.$$

Substituting b for x, we get the follwoing equation in the algebra A:

$$(\lambda^{-1}b)^n = \lambda_1^{-1}b \odot \lambda_2^{-1}b \odot \cdots \odot \lambda_n^{-1}b.$$

Since each $\lambda_k^{-1}b$ is $\odot$-invertible, so is $(\lambda^{-1}b)^n$. Let

$$c_k = -\sum_{j=1}^{n-1} \lambda_k^{-j} b^{-j} = -\lambda_k^{-1}b - \lambda_k^{-2}b^2 - \cdots - \lambda_k^{1-n}b^{1-n}.$$

Then

$$(\lambda^{-1}b)^n = \lambda_k^{-1}b \odot c_k,$$

so,

$$(\lambda_k^{-1}b)^\circ = c_k \odot ((\lambda^{-1}b)^n)^\circ.$$

If $\alpha^n = 1$, then

$$\sum_{k=1}^{n} \alpha^k = \frac{\alpha - \alpha^{n+1}}{1-\alpha} = \frac{\alpha - \alpha}{1-\alpha} = 0, \quad \text{so} \quad \sum_{k=1}^{n} c_k = 0.$$

Thus,

$$\sum_{k=1}^{n} (\lambda_k^{-1}b)^\circ = \sum_{k=1}^{n} (c_k \odot ((\lambda^{-1}b)^n)^\circ) = n((\lambda^{-1}b)^n)^\circ,$$

so

$$((\lambda^{-1}b)^n)^\circ = \frac{1}{n} \sum_{k=1}^{n} (\lambda_k^{-1}b)^\circ.$$

Now,

$$\left\| ((\lambda^{-1}b)^n)^\circ - ((r^{-1}b)^n)^\circ \right\| \le \frac{1}{n} \sum_{k=1}^{n} \left\| (\lambda_k^{-1}b)^\circ - (r_k^{-1}b)^\circ \right\|.$$

By uniform continuity, for any $\varepsilon > 0$, we can pick $\lambda > r$ with

$$\left\| ((\alpha\lambda)^{-1}b)^\circ - (\alpha r^{-1}b)^\circ \right\| \le \varepsilon, \quad \text{for all } \alpha \text{ with } |\alpha| = 1.$$

Then

$$\left\| ((\lambda^{-1}b)^n)^\circ - ((r^{-1}b)^n)^\circ \right\| \le \varepsilon.$$

But,

$$\left\| (\lambda^{-1}b)^n \right\| \le (|\lambda^{-1}| \|b^n\|^{\frac{1}{n}})^n \to (\lambda^{-1}r)^n \to 0, \quad \text{so} \quad ((\lambda^{-1}b)^n)^\circ \to \mathbf{0}.$$

Thus,

$$\left\| ((r^{-1}b)^n)^\circ \right\| \to 0, \quad \text{so} \quad (r^{-1}b)^n \to \mathbf{0}.$$

But this contratricts

$$\left\| (r^{-1}b)^n \right\| = r^{-n} \|b^n\| \ge \|b^n\|^{-1} \|b^n\| = 1.$$

$\square$

Corollary 4.17. *If A is a Banach algebra, then for any $b \in A$,*

$$|b|_\sigma = \inf_n \|b^n\|^{\frac{1}{n}}.$$

Note that $|b|_\sigma$ only depends on the algebra structure of A and $\inf_n \|b^n\|^{\frac{1}{n}}$ only depends on the norm and product structure.

Corollary 4.18. *If A is a C^*-algebra, then for any self-adjoint $b \in A$,*

$$|b|_\sigma = \|b\|.$$

Proof. If $b = b^*$, then $\left\|b^2\right\| = \|b^*b\| = \|b\|^2$. By induction, $\left\|b^{2^n}\right\| = \|b\|^{2^n}$. Thus,

$$\|b\| = \left\|b^{2^n}\right\|^{\frac{1}{2^n}} \to |b|_\sigma,$$

by Corollary 4.17. $\square$

Corollary 4.19. *If A and B are C^*-algebras and $\pi\colon A \to B$ is a *-algebra homomorphism, then*

$$\|\pi\| \leq 1.$$

Proof. If $\lambda^{-1}a$ is $\odot$-invertible, then so is $\pi(\lambda^{-1}a) = \lambda^{-1}\pi(a)$. Thus, $\sigma(\pi(a)) \subseteq \sigma(a)$ and hence $|\pi(a)|_\sigma \leq |a|_\sigma$. If a is self-adjoint, then so is $\pi(a)$ and hence $\|\pi(a)\| \leq \|a\|$, by Corollary 4.18. For general $a \in A$, $\|\pi(a)\| = \sqrt{\|\pi(a^*a)\|} \leq \sqrt{\|a^*a\|} = \|a\|$. $\square$

Corollary 4.20. *C^*-algebra norms are unqiue.*

Proof. If $\|\cdot\|$ and $\|\cdot\|'$ are C*-algebra norms on a *-algebra A, taking $\pi = \mathrm{id}$ and applying Corollary 4.19, we get $\|a\| \leq \|a\|' \leq \|a\|$, for all $a \in A$. $\square$

5 Representations

Spectral theory will be applied to obtain representations. Recall that, for any set X, we have commutative C*-algebra

$$\ell^\infty(X) = \{f \in \mathbb{C}^X : \|f\| < \infty\},$$

where $\|f\| = \sup_{x\in X}|f(x)|$. $\ell^\infty(X)$ also has many C*-subalgebras, e.g., any topology on X determines C*-subalgebra of continuous bounded functions $C^b(X)$. Given $x \in X$, we have a further (potentially non-unital) C*-subalgebra

$$A = \left\{f \in C^b(X) : f(x) = 0\right\}.$$

It turns out that all commutative C*-algebras can be *represented* as C*-algebra of continuous bounded functions on a topological space, vanishing at a particular point in the non-unital case.

Basic idea. Given a C*-algebra A, we can consider *characters* on A:

$$\Phi_A = \{\psi \in \mathbb{C}^A : \psi \text{ is a (*-)algebra homomorphism}\}.$$

Every $b \in A$ determines a function $\widehat{b}\colon \Phi_A \to \mathbb{C}$, defined by

$$\widehat{b}(\psi) = \psi(b).$$

Then $b \mapsto \widehat{b}$ is a homomorphism $A \to \ell^\infty(\Phi_A)$:

- $\widehat{ab} = \widehat{a}\widehat{b}$ because $\widehat{ab}(\psi) = \psi(ab) = \psi(a)\psi(b) = \widehat{a}(\psi)\widehat{b}(\psi) = (\widehat{a}\widehat{b})(\psi)$.
- Similarly, $\widehat{a+b} = \widehat{a} + \widehat{b}$, $\widehat{b^*} = \widehat{b}^*$, and $\widehat{\lambda b} = \lambda\widehat{b}$.
- $\|\widehat{b}\| \leq \widehat{b}$ $(\leq \infty)$ because $|\widehat{b}(\psi)| = |\psi(b)| \leq \|\psi\|\|b\| \leq \|b\|$.

However, $b \mapsto \widehat{b}$ may not be an isomorphism. For example, if $A = M_2(\mathbb{C})$, then $\Phi_A = \{0\}$, so $\widehat{b} = 0$, for all $b \in A$. *The goal* is to show that $\|\widehat{b}\| = \|b\|$, for all $b \in A$, when A is commutative.

5.1 Gelfand representation

It will be convenient to add a unit to any non-unital algebra A. Algebraically this is simple: let $A' = A \times \mathbb{C}$ with

$$\begin{aligned}(a,\lambda)^* &= (a^*, \overline{\lambda}),\\ \lambda(a,\gamma) &= (\lambda a, \lambda\gamma),\\ (a,\lambda)+(b,\gamma) &= (a+b, \lambda+\gamma),\\ (a,\lambda)(b,\gamma) &= (ab+\gamma a+\lambda b, \lambda\gamma).\end{aligned}$$

Then A' is a *-algebra with *-subalgebra $A \times \{0\}$ isomorphic to A. But, A' also has a unit $(\mathbf{0}, 1)$.

Defining a norm to make A' a C*-algebra is more tricky. One could define *-algebra norm by $\|(a,\lambda)\| = \|a\| + |\lambda|$, but this norm might not satisfy the C*-condition, namely it might happen that $\|(a,\lambda)^*(a,\lambda)\| \neq \|(a,\lambda)\|^2$.

Theorem 5.1. *A' is a C*-algebra under the norm*

$$\|(a,\lambda)\| = \sup_{\|b\|\leq 1} \|ab+\lambda b\|.$$

Proof. Note that this is the operator norm, considering $(a,\lambda) \in \mathcal{B}(A)$, defined by

$$(a,\lambda)(b) = ab + \lambda b.$$

This makes sense also algebraically with how the operations on A were extended above and also $\|(a,0)\| = \|a\|$. Thus, the norm is an algebra norm. Also, 1 dimensional extensions of Banach spaces are always complete. So we only need to check the C* norm condition $\|x\|^2 \leq \|x^*x\|$.

For any $(a,\lambda) \in A'$,

$$\begin{aligned}\|(a,\lambda)\|^2 &= \sup_{\|b\|\leq 1} \|(a,\lambda)(b)\|^2 = \sup_{\|b\|\leq 1} \|ab+\lambda b\|^2\\ &= \sup_{\|b\|\leq 1} \left\|(b^*a^* + \overline{\lambda}b^*)(ab+\lambda b)\right\|\\ &= \sup_{\|b\|\leq 1} \left\|b^*[(a^*,\overline{\lambda})\circ(a,\lambda)](b)\right\|\\ &= \sup_{\|b\|\leq 1} \left\|[(a^*,\overline{\lambda})\circ(a,\lambda)](b)\right\|\\ &= \left\|(a^*,\overline{\lambda})(a,\lambda)\right\|\\ &= \|(a,\lambda)^*(a,\lambda)\|\end{aligned}$$

Thus, the C* norm condition holds. $\square$

Proposition 5.2. *Every character on a Banach algebra is automatically continuous.*

Proof. Consider a character, i.e., an algebra homomorphism $\phi\colon A \to \mathbb{C}$. For any $a \in A$, $\{\phi(a)\} = \sigma(\phi(a)) \subseteq \sigma(a)$. Thus $|\phi(a)| \leq |a|_\sigma \leq \|a\|$, i.e., $\|\phi\| \leq 1$. $\square$

Proposition 5.3. *If ϕ is a character on $A \subseteq A' = A + \mathbb{C}\mathbf{1}$, then*

$$\phi'(a+\lambda\mathbf{1}) = \phi(a) + \lambda$$

defines a character ϕ' on A' extending ϕ on A.

Theorem 5.4. *Characters take real values on self-adjoint elements of C^*-algebras.*

Proof. Let ϕ be a character on A. Extend ϕ to $A' = A + \mathbb{C}\mathbf{1}$. Given $a \in A$ and $t \in \mathbb{R}$, define

$$e^{ita} = \sum_{n=0}^{\infty} \frac{1}{n!}(ita)^n.$$

Thanks to completeness, this is always well-defined. If $a = a^*$, then

$$(e^{ita})^* = \sum_{n=0}^{\infty} \frac{1}{n!}(-ita)^n = e^{-ita}.$$

Thus,

$$\left\|e^{ita}\right\|^2 = \left\|(e^{ita})^* e^{ita}\right\| = \left\|e^{-ita}e^{ita}\right\| = \left\|e^{\mathbf{0}}\right\| = \|\mathbf{1}\| = 1.$$

So,

$$1 \geq \|\phi\| \geq |\phi(e^{ita})| = |e^{it\phi(a)}| = e^{-tr},$$

where $\phi(a) = s + ri$. If $r \neq 0$, then for large positive or negative t this would be impossible. Thus, $\phi(a) = s \in \mathbb{R}$. $\square$

Corollary 5.5. *Characters on C^*-algebras are automatically $*$-homomorphisms.*

Proof. For any $a \in A$, $b = \frac{1}{2}(a^* + a)$ and $c = \frac{1}{2}i(a^* - a)$ are sefl-adjoint. Moreover, $b + ic = \frac{1}{2}(a^* + a) - \frac{1}{2}(a^* - a) = a$. Using Theorem 5.4, if ϕ is a character, then

$$\phi(a^*) = \phi(b - ic) = \phi(b) - i\phi(c) = \overline{\phi(b) + i\phi(c)} = \overline{\phi(b + ic)} = \overline{\phi(a)}.$$

$\square$

Definition 5.6. An *ideal* in an algebra is a subspace I such that, for all $b \in A$,

$$bI \cup Ib \subseteq I.$$

Given an ideal I, we can form the quotient algebra A/I. The elements of A/I are subsets of the form $b + I$, for $b \in A$. We define $(a + I)(b + I) = ab + I$. This is well-defined since, for all $j, k \in I$,

$$(a + j)(b + k) + I = ab + ak + jb + jk + I = ab + I.$$

Likewise, define $(a + i) + (b + I) = a + b + I$ and $\lambda(a + I) = \lambda a + I$. If A is a normed algebra and I is a closed ideal, then defining

$$\|a + I\| = \inf_{j \in I} \|a + j\|$$

turns A/I into a normed algebra as well.

Definition 5.7. An ideal I in an algebra A is *modular* if A/I is unital.

Proposition 5.8. *If A is a commutative algebra, then any $e \in A$ defines a modular ideal*

$$I_e = \{a - ae : a \in A\}.$$

Proof. For any $b \in I_e$ and $c \in A$, we have $a \in A$ with $b = a - ae$, so

$$bc = cb = c(a - ae) = (ca) + (ca)e \in I_e.$$

Likewise, we can verify I_e to be a subspace and hence an ideal. Moreover,

$$(e + I_e)(c + I_e) = ec + I_e = c - c + ec + I_e = c + I_e.$$

Thus, $e + I_e$ is a unit of A/I_e and hence I_e is modular. □

Proposition 5.9. *If I is an ideal and $e + I$ is a unit of A/I, then*

$$e \text{ is } \odot\text{-invertible} \quad \Longrightarrow \quad e \in I \iff A = I.$$

(The $\Longrightarrow$ can be reversed when $I = I_e$.)

Proof. Note that $I_e \subseteq I$ since

$$(ae + I) = (a + I)(e + I) = (a + I),$$

so $a - ae \in I$.

Suppose that e has $\odot$-inverse e°, so $\mathbf{0} = e^\circ \odot e = e^\circ + e + e^\circ e$. Thus, $e = e^\circ e - e^\circ \in I_e \subseteq I$, proving the first $\Rightarrow$.

Now, $e \in I$ implies $ae \in I$, for any $a \in A$. Then $a = a - ae + ae \in I_e + I \subseteq I$, proving the second $\Rightarrow$.

Conversely, if $A = I$, then, in particular, $e \in I$, giving the last $\Leftarrow$. □

Proposition 5.10. *Every proper modular ideal has maximal extension.*

Proof. Suppose that I is a proper ideal and $e + I$ is a unit of A/I, for some $e \in A$. Thus $e \notin I$, by the previous result. By Kuratowski-Zorn, we have a maximal ideal $J \supseteq I$ with $e \notin J$. So any ideal extension K of J would contain e. As $ae - a \in I \subseteq J \subseteq K$, $e + K$ would again be a unit of A/K:

$$(e + K)(a + K) = ae + K = a - a + ae + K = a + K.$$

The previous result would imply $K = A$. This show that J is a maximal proper ideal. Why cant we apply Kuratowski-Zorn directly to proper ideals? □

Proposition 5.11. *If A is a normed algebra, I is a proper ideal and $e + I$ is a unit of A/I, then*

$$\inf_{j \in I} \|e - j\| \geq 1.$$

Proof. Say we had $j \in I$ with $\|e - j\| < 1$. Then $e - j$ is $\odot$-invertible, so we can define $f = (e - j)^\circ$. This means that

$$\mathbf{0} = f \odot (e - j) = f + e - j - fe + fj = \mathbf{0}, \quad \text{so} \quad e = j - f + fe - fj \in I + I_e - I \subseteq I.$$

But then $A = I$, contradicting properness. □

Corollary 5.12. *Every maximal modular ideal in a normed algebra is closed.*

Corollary 5.13. *If A is commutative normed algebra and I is a maximal modular ideal, then*

$$A/I \cong \mathbb{C}.$$

Proof. Since I is closed, $B = A/I$ is a commutative normed algebra with unit e. B has no non-zero proper ideals: if J were a non-zero proper ideal of B, $\bigcup J$ would be an ideal of A with $I \subsetneq \bigcup J \neq A$, a contradiction. Thus, for all $c \in B \setminus \{\mathbf{0}\}$, $Bc = B$, i.e., c is invertible. Since $\sigma(c) \neq \emptyset$, we have $\lambda \in \mathbb{C}$ with $\lambda e - c$ not invertible, so $c = \lambda e$. This show that $B \cong \mathbb{C}e \cong \mathbb{C}$. □

Corollary 5.14. *Maximal modular ideals in a commutative normed algebra A are precisely the kernels* $\operatorname{Ker}(\psi) = \{a \in A : \psi(a) = 0\}$ *of non-zero characters $\psi \in \Phi_A$.*

Theorem 5.15 (Gelfand isomorphism)**.** *If A is a commutative C^*-algebra, then*

$$\|\widehat{a}\| = \|a\|,$$

for all $a \in A$.

Proof. We already saw that $\|\widehat{a}\| \le \|a\|$. Conversely, first, take a self-adjoint $a \in A$. Then $\|a\| = |a|_\sigma$ by Corollary 4.18. So, we have $\lambda \in \sigma(a)$ with $|\lambda| = \|a\|$. Thus, $\lambda^{-1}a$ is not $\odot$-invertible. By Proposition 5.8, $I_{\lambda^{-1}a}$ is a modular ideal. By Proposition 5.9, $I_{\lambda^{-1}a}$ is a proper modular ideal. By Proposition 5.10, we can extend $I_{\lambda^{-1}a}$ to a maximal modular ideal M which is by Corollary 5.14 the kernel of a non-zero character $\psi \in \Phi_A$. Since $\lambda^{-1}a + M$ is a unit for A/M, $\psi(\lambda^{-1}a) = 1$. Thus $\widehat{a}(\psi) = \psi(a) = \lambda$ and hence $\|\widehat{a}\| \ge |\lambda| = \|a\|$.

For general $a \in A$, we have

$$\|a\|^2 = \|a^*a\| = \|\widehat{a^*a}\| = \|\widehat{a}^*\widehat{a}\| = \|\widehat{a}\|^2.$$

□

So, we can identify A with $\{\widehat{a} : a \in A\} \subseteq \ell^\infty(\Phi_A)$. The next goal is to turn Φ_A into a compact topological space so that A can be identified precisely with the continuous functions from $\Phi_A \to \mathbb{C}$.

Definition 5.16. A *topology* $\mathcal{O}(X)$ on a set X is a family of *open* subsets of X such that

$$\begin{aligned} \emptyset \in \mathcal{O}(X), &\quad X \in \mathcal{O}(X), \\ O, N \in \mathcal{O}(X) &\implies O \cap N \in \mathcal{O}(X), \\ (O_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{O}(X) &\implies \bigcup_\lambda O_\lambda \in \mathcal{O}(X). \end{aligned}$$

We call $\mathcal{B} \subseteq \mathcal{O}(X)$ a *basis* if every open $\mathcal{O}$ is a union from $\mathcal{B}$, i.e.,

$$\mathcal{O}(X) = \left\{ \bigcup_\lambda B_\lambda : (B_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{B} \right\}.$$

We call $\mathcal{S} \subseteq \mathcal{O}(X)$ a *subbasis* if the finite intersections from $\mathcal{S}$

$$\left\{ \bigcap \mathcal{F} : \mathcal{F} \subseteq \mathcal{S} \text{ is finite} \right\}$$

form a basis.

Example 5.17. Every metric defines a canonical topology. Specifically, if d is a metric on X, we define $O \subseteq X$ to be open if for every $x \in O$ there is $\varepsilon > 0$ such that for every $y \in X$, $d(x, y) < \varepsilon$ implies that $y \in O$. This topology has basis $\mathcal{B}$ of balls $B_x^\varepsilon = \{y \in X : d(x, y) < \varepsilon\}$, for $x \in X$ and $\varepsilon > 0$.

For example, if d is the usual metric on $\mathbb{R}$, given by $d(x, y) = |x - y|$, then

$$\mathcal{O}(X) = \left\{ \bigcup_{n\in\mathbb{N}} (a_n, b_n) : (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \right\}.$$

The ball basis is $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$. A subbasis is $\mathcal{S} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$.

Definition 5.18. A map $f : X \to Y$ between topological spaces is *continuous* if

$$N \in \mathcal{O}(Y) \implies f^{-1}(N) \in \mathcal{O}(X).$$

This agrees with the metric notion of continuity. Also, it suffices to consider N is a basis or a subbasis. A composition of continuous maps is also continuous.

Definition 5.19. A topological space X is *compact* if every open cover has a finite subcover. In other words, for any open family $(O_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{O}(X)$,

$$X = \bigcup_{\lambda\in\Lambda} O_\lambda \implies \exists \text{ finite } F \subseteq \Lambda \text{ such that } X = \bigcup_{\lambda\in F} O_\lambda.$$

Lemma 5.20. *Continuous image of a compact space X is compact and any* closed *subset $C \subseteq X$, that is $X \setminus C$ is open, is also compact.*

Lemma 5.21. *For compactness, it suffices to consider covers from a basis $\mathcal{B}$*

Proof. We prove it for basis $\mathcal{B}$. Given an open cover $\mathcal{C} \subseteq \mathcal{O}(X)$, let $\mathcal{C}' = \{B \in \mathcal{B} : B \subseteq O \in \mathcal{C}\}$. Since $\mathcal{B}$ is a basis, $\mathcal{C}'$ still covers X. By assumption, we have $B_1, \ldots, B_n$ covering X. Thus, we have $O_1 \supseteq B_1, \ldots, O_n \supseteq B_n$ from $\mathcal{C}$ covering X. □

Theorem 5.22 (Alexander subbasis theorem)**.** *If every cover from a subbasis $\mathcal{S}$ has a finite subcover, then X is compact.*

Proof. Suppose that $\mathcal{C} \subseteq \mathcal{O}(X)$ covers X but has no finite subcover. By Kuratowski-Zorn, we can take maximal $\mathcal{C}$ with this property. By assumption $\mathcal{C} \cap \mathcal{S}$ can not cover X. Take $x \in X \setminus \bigcup(\mathcal{C} \cap \mathcal{S})$ and $O \in \mathcal{C}$ with $x \in O$. Since $\mathcal{S}$ is subbasis, we have finite $\mathcal{F} \subseteq \mathcal{S}$ with $x \in \bigcap \mathcal{F} \subseteq O$. By the choice of x, it must be that $\mathcal{F} \cap \mathcal{C} = \emptyset$. By the maximality, for $S \in \mathcal{F}$, we have finite $\mathcal{G}_S \subseteq \mathcal{C}$ with $X = S \cup \bigcup \mathcal{G}_S$. Thus,

$$X = \bigcap \mathcal{F} \cup \bigcup \mathcal{G} = O \cup \bigcup \mathcal{G}, \quad \text{where} \quad \mathcal{G} = \bigcup_{S\in\mathcal{F}} \mathcal{G}_S.$$

So $\{O\} \cup \mathcal{G}$ is a finite cover of X from $\mathcal{C}$, a contradiction. □

Definition 5.23. A *subspace* is a subset Y of a space X with the subspace topology

$$\mathcal{O}(Y) = \{N \cap Y : N \in \mathcal{O}(X)\}.$$

Example 5.24. The unit interval $[0,1]$, as a subspace of $\mathbb{R}$, has the subbasis

$$\mathcal{S} = \{\emptyset, [0,1]\} \cup \{(a,1] : 0 < a < 1\} \cup \{[0,b) : 0 < b < 1\}.$$

Theorem 5.25. *The unit interval $[0,1]$ is a compact topological space.*

Proof. By Theorem 5.22, it suffices to consider the subbases. For contradiction, suppose that $\mathcal{C} \subseteq \mathcal{S}$ has no finite subcover. We need to show that $\mathcal{C}$ is not a cover.

- If $\mathcal{C} \subseteq \{(a,1] : 0 < a < 1\}$, then $0 \notin \bigcup \mathcal{C}$, so $X \neq \bigcup \mathcal{C}$.
- If $\mathcal{C} \subseteq \{[0,b) : 0 < b < 1\}$, then $1 \notin \bigcup \mathcal{C}$, so $X \neq \bigcup \mathcal{C}$.

Otherwise,

$$b' = \sup\{b : [0,b) \in \mathcal{C}\} \leq \inf\{a : (a,1] \in \mathcal{C}\} = a'.$$

Then $x \notin \bigcup \mathcal{C}$, for any x with $b' \leq x \leq a'$, so again $X \neq \bigcup \mathcal{C}$. □

Given topological spaces $(X_\lambda)_{\lambda \in \Lambda}$, consider their product

$$Y = \prod_{\lambda \in \Lambda} X_\lambda = \{(y_\lambda)_{\lambda \in \Lambda} : \forall \lambda \in \Lambda \ (y_\lambda \in X_\lambda)\}.$$

For each $\gamma \in \Lambda$, we have a projection $p_\gamma \colon Y \to X_\gamma$ onto the coordinate γ:

$$p_\gamma((y_\lambda)_{\lambda \in \Lambda}) = y_\gamma.$$

We take the cylinder sets

$$p_\gamma^{-1}(O) = \{(y_\gamma) \in Y : y_\gamma \in O\},$$

for $\gamma \in \Lambda$ and $O \in \mathcal{O}(X_\gamma)$, as a subbasis for a topology on Y. So, $O \in \mathcal{O}(Y)$ if and only if O is a union of finite cylinder intersections

$$p_{\gamma_1}^{-1}(O_1) \cap \cdots \cap p_{\gamma_n}^{-1}(O_n).$$

This is the coarsest topology making all the projections continuous.

Theorem 5.26 (Tikhonov's theorem)**.** *If $(X_\lambda)_{\lambda \in \Lambda}$ are all compact spaces, then $Y = \prod_{\lambda \in \Lambda} X_\lambda$ is also compact.*

Proof. By Theorem 5.22, we only need to consider cylinder sets $p_\lambda^{-1}(O)$. Let $\mathcal{C}$ be a collection of cylinder sets with no finite subcollection covering Y. We need to show the entirety of $\mathcal{C}$ does not cover Y. For each $\lambda \in \Lambda$, let $\mathcal{C}_\lambda = \left\{O \in \mathcal{O}(X_\lambda) : p_\lambda^{-1}(O) \in \mathcal{C}\right\}$. Not that $\mathcal{C}_\lambda$ does not cover X_λ – otherwise it would have a finite subcover and the corresponding cylinder sets would cover Y. So, we may pick $y_\lambda \in X_\lambda \setminus \bigcup \mathcal{C}_\lambda$, for each $\lambda \in \Lambda$. This gives us $(y_\lambda)_{\lambda \in \Lambda} \in Y$ not covered by any $\mathcal{C}_\lambda$. Thus $\mathcal{C} = \bigcup_{\lambda \in \Lambda} \mathcal{C}_\lambda$ is not a cover of Y. □

Proposition 5.27. *Every normed algebra A has compact character space Φ_A.*

Proof. Due to linearity, each character $\psi \in \Phi_A$ on a normed algebra A is determined by its values on the unit ball $A^1 = \{a \in A : \|a\| \leq 1\}$. Also, $\|\psi\| \leq 1$, so $\psi \in \Phi_A$ maps A^1 to $\mathbb{C}^1$. Since $\mathbb{C}^1$ is compact, so is $Y = \prod_{a \in A^1} \mathbb{C}^1$, by Theorem 5.26. Now, Φ_A is the intersection of the following, for $a, b \in A$ and $t \in (0,1)$:

$$
\begin{aligned}
C_{t,a,b} &= \{\psi \in Y : \psi(ta + (1-t)b) - t\psi(a) - (1-t)\psi(b) = 0\}, \\
M_{a,b} &= \{\psi \in Y : \psi(ab) - \psi(a) - \psi(b) = 0\}.
\end{aligned}
$$

Each set here is closed, hence their intersection Φ_A is also closed. Since Φ_A is a closed subset of compact Y, Φ_A is also compact. □

Note that $\widehat{a}(\psi) = \psi(a)$ is just the projection of ψ onto "coordinate a". Thus, Gelfand trasnfrom $\widehat{a} \colon \Phi_A \to \mathbb{C}$ of a is continuous on Φ_A. So, we can identify any commutative C*-algebra A with the C*-subalgebra $\{\widehat{a} : a \in A\}$ of continuous functions from $\Phi_A \to \mathbb{C}$. Does this C*-subalgebra contain all continuous functions $C(\Phi_A)$? Not quite – the zero homomorphism $0 \in \Phi_A$ always gets mapped to 0 by any Gelfand transform $\widehat{a}$. But this is really the only restriction. We show this by approximating any $f \in C(\Phi_A)$ with $f(0) = 0$ by $(\widehat{a_n})$. By completeness, we have $a_n \to a \in A$ and hence $f = \widehat{a}$.

First, we show that f can at least be approximated on pairs of points. This will rely on the following result, showing that characters are determined already by their scalar multiples.

Proposition 5.28. *If $\phi, \psi \in \Phi_A \setminus \{0\}$ and $\phi = \lambda\psi$, for some $\lambda \in \mathbb{C}$, then $\phi = \psi$.*

Proof. Since $\psi \neq 0$, we have $a \in A$ with $\psi(a) \neq 0$. Note $\lambda^2\psi(a)^2 = \phi(a)^2 = \phi(a^2) = \lambda\psi(a^2) = \lambda\psi(a)^2$. Since $\psi(a) \neq 0$, this gives $\lambda^2 = \lambda$ and hence $\lambda = 0$ or $\lambda = 1$. Since $\phi \neq 0$, we must have $\lambda = 1$, so $\phi = \psi$. □

Proposition 5.29. *For distinct $\phi, \psi \in \Phi_A \setminus \{0\}$ we have $a \in A$ with $\phi(a) = 1$ and $\psi(a) = 0$.*

Proof. Take $b \in A$ with $\phi(b) \neq 0$. By Proposition 5.28,

$$\psi \neq \frac{\psi(b)}{\phi(b)}\phi.$$

So we have $c \in A$ with

$$\phi(b)\psi(c) \neq \psi(b)\phi(c).$$

Thus,

$$\phi(\psi(c)b - \psi(b)c) = \phi(b)\psi(c) - \psi(b)\phi(c) \neq 0$$

and

$$\psi(\psi(c)b - \psi(b)c) = \psi(b)\psi(c) - \psi(b)\psi(c) = 0.$$

Taking

$$a = \frac{\psi(c)b - \psi(b)c}{\phi(b)\psi(c) - \psi(b)\phi(c)}$$

gives $\phi(a) = 1$ while $\psi(a) = 0$. □

Corollary 5.30. *For distinct $\phi, \psi \in \Phi_A \setminus \{0\}$ and $\alpha, \beta \in \mathbb{C}$, we have $a \in A$ with*

$$\phi(a) = \alpha \quad \textit{and} \quad \psi(a) = \beta.$$

The next step is to build up from points to the whole space using compactness. This is done via the Stone-Wierstrass theorem.

Theorem 5.31 (Stone-Weirstrass theorem)**.** *Suppose that $A \subseteq C(X, \mathbb{R})$ is closed under $\vee$ and $\wedge$, defined by*

$$(a \vee b)(x) = \max(a(x), b(x)),$$
$$(a \wedge b)(x) = \min(a(x), b(x)).$$

If X is compact and $f \in C(X, \mathbb{R})$, then

$$\forall y, z \in X \quad \inf_{a \in A}(|f(y) - a(y)| + |f(z) - a(z)|) = 0 \quad \implies \quad \inf_{a \in A} \|f - a\| = 0.$$

Proof. For every $\varepsilon > 0$ and $y, z \in X$, we have $a_{yz} \in A$ with

$$|f(y) - a_{yz}(y)| < \varepsilon \quad \text{and} \quad |f(z) - a_{yz}(z)| < \varepsilon.$$

In other words, both y and z are in the open subsets

$$U_{yz} = \{x \in X : f(x) - a_{yz}(x) < \varepsilon\},$$
$$V_{yz} = \{x \in X : a_{yz}(x) - f(x) < \varepsilon\}.$$

For a fixed y, compactness gives finite $G \subseteq X$ with $X = \bigcup_{z \in G} U_{yz}$. Taking $a_y = \bigvee_{z \in G} a_{yz} \in A$ gives $f(x) - a_y(x) < \varepsilon$, for all $x \in X$. Also $a_y(x) - f(x) < \varepsilon$, for all $x \in V_y = \bigcap_{z \in G} V_{yz}$. Again compactness gives finite $F \subseteq X$ with $X = \bigcup_{y \in F} V_y$. Taking $a = \bigwedge_{y \in F} a_y \in A$ gives $a(x) - f(x) < \varepsilon$, for all $x \in X$. But still $f(x) - a(x) < \varepsilon$, for all $x \in X$, as this holds for each a_y. Thus, $\|f - a\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\inf_{a \in A} \|f - a\| = 0$. □

Theorem 5.32. *If $A \subseteq C(X, \mathbb{R})$ is a real Banach algebra, A is closed under $\vee$ and $\wedge$.*

Proof. We claim $\sqrt{a} \in A$, for any $a \in A_+ = C(X, \mathbb{R}_+)$. To see this, note that $\sqrt{r}$ on $\mathbb{R}_+$ can be approximated uniformely by polynomials with zero constant term on $[0, s]$, for any $s > 0$:

- There is an analytic branch of $\sqrt{\alpha + \beta i}$ defined on $\mathbb{C} \setminus (-\mathbb{R}_+)$.
- Thus, the power series centered at s converges uniformely on $[\delta, 2s - \delta]$.
- So, we have a polynomial p with $|p(r) - \sqrt{r}| < \varepsilon$, for $r \in [\delta, 2s - \delta]$.
- Then $q(r) = p(r+\delta) - p(\delta)$ is a polynomial with zero constant term such that $|q(r) - \sqrt{r}| < \varepsilon'$, for all $r \in [0, 2s - 2\delta] \supseteq [0, s]$.

Taking $s = \|a\|$, $q(a) \in A$ and $\|q(a) - \sqrt{a}\| < \varepsilon$. Since ε was arbitrary and A is complete and closed, $\sqrt{a} \in A$.

Thus, for all $a \in A$, $|a| = \sqrt{a^2} \in A$ and hence, for all $a, b \in A$,

$$a \vee b = \frac{1}{2}(a + b + |a - b|) \in A \quad \text{and} \quad a \wedge b = \frac{1}{2}(a + b - |a - b|) \in A.$$

□

Theorem 5.33. *If A is a commutative C^*-algebra, then*

$$\{\widehat{a} : a \in A\} = C_0(\Phi_A).$$

Proof. Note that $\{\widehat{a} : a = a^*\} \subseteq C(\Phi_A, \mathbb{R})$ is a real Banach algebra. By Theorem 5.32, it is closed under $\vee$ and $\wedge$. By Corollary 5.30, for any $f \in C(\Phi_A, \mathbb{R})$ and $\phi, \psi \in \Phi_A \setminus \{0\}$, we have $a \in A$ with $\widehat{a}(\phi) = f(\phi)$ and $\widehat{a}(\psi) = f(\psi)$. Since f is real, we can ensure that $a = a^*$. By Theorem 5.31, as long as $f(0) = 0$, we have $(a_n) \subseteq A$ with $a_n = a_n^*$ and $\|\widehat{a}_n - f\| \to 0$. By completeness, $a_n \to a$, so $f = \widehat{a}$.

Since every $f \in C(\Phi_A)$ can be written as $f = g + ih$, for $g, h \in C(\Phi_A, \mathbb{R})$, it follows that $f = \widehat{b} + i\widehat{c} = \widehat{a}$, for $a = b + ic \in A$. □

Thus, all commutative C*-algebras are isomorphic to $C_0(X)$, for some compact space X and some fixed point $0 \in X$.

Proposition 5.34. *If A is unital, then the zero character 0 is isolated (that is open) in Φ_A.*

Proof. Note that $\psi(1) = 1$, for $\psi \in \Phi_A \setminus \{0\}$ while $0(1) = 0$. Thus,

$$\{0\} = \widehat{1}^{-1}\left(\left\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2}\right\}\right)$$

is open. □

This means that $\Phi_A \setminus \{0\}$ is also compact and A is isomorphic to $C(\Phi_A)$. In other words, all unital commutative C*-algebras are isomorphic to those of the form $C(X)$, for some compact space X.

Even in non-commutative C*-algebras, the Gelfand representation gives a powerful "continuous functional calculus" on normal elements. First, ote that any $b \in A$ *generates* a C*-algebra B given by

$$B = \mathrm{cl}(\{p(b) : p \text{ is a *-polynomial with no constant term}\}).$$

Equivalently, B is the smallest C*-subalgebra of A containing b. If b is normal, i.e., $b^*b = bb^*$, then $p(b)q(b) = q(b)p(b)$, for all *-polynomials p and q, and hence B is commutative. Thus, by Gelfand, we can identify B with $C_0(\Phi_B)$. For any $f \in C_0(\mathbb{C})$, not that $f \circ \widehat{b} \in C_0(\Phi_B)$. Thus, we have some $f(b) \in B$ with $\widehat{f(b)} = f \circ \widehat{b}$, i.e., for all $\psi \in \Phi_B$,

$$\psi(f(b)) = f(\psi(b)).$$

In fact, we really only need f to be defined on $\sigma(b)$.

Recall the definition of the spectrum of an element b of an algebra A:

$$\sigma_A(b) \setminus \{0\} = \big\{\lambda \in \mathbb{C} \setminus \{0\} : k\lambda^{-1}b \text{ is not } \odot\text{-invertible in } A\big\}.$$

The ambient algebra A is crucial. If b is in a subalgebra B of A and $\lambda^{-1}b$ has no $\odot$-inverse in A, then it centrainly can not have $\odot$-inverse in B either, so

$$\sigma_A(b) \setminus \{0\} \subseteq \sigma_B(b) \setminus \{0\}.$$

In general (even Banach) algebras, this inclusion can be strict. However, for C*-(sub)algebras, we always have equality. For simplicity, we restrict ourselves to normal elements.

Theorem 5.35. *If B is a C*-subalgebra of A and $b \in B$ is normal, then*

$$\sigma_A(b) \setminus \{0\} = \sigma_B(b) \setminus \{0\}.$$

Proof. Note it suffices to take B as the C*-subalgebra generated by B. Let $f_n \in [0,1]^{\mathbb{C}}$ be "bump functions" at 1 with bumps of radius $1/n$:

$$f_n(\alpha) = 0 \vee (1 - n|1-\alpha|).$$

Given $\lambda \in \sigma_B(b) \setminus \{0\}$, let $a_n = f_n(\lambda^{-1}b)$ so $\|a_n\| = 1$ and

$$\left\|a_n - a_n\lambda^{-1}b\right\| \to 0.$$

Thus

$$a_n \odot \lambda^{-1}b = \lambda^{-1}b + a_n - a_n\lambda^{-1}b \to \lambda^{-1}b.$$

If $\lambda^{-1}b$ had a $\odot$-inverse c in A, we would have a contradiction:

$$a_n = a_n \odot \lambda^{-1}b \odot c \to \lambda^{-1}b \odot c = 0.$$

$\square$

Recall that for any function f in algebra $C_0(X)$,

$$\mathrm{ran}(f) = \sigma(f) \cup \{0\}.$$

Indeed, for $\lambda \notin \sigma(f) \cup \{0\}$, the $\odot$-inverse of $\lambda^{-1}f$ is given by

$$g(x) = \frac{f(x)}{f(x) - \lambda}.$$

If $b \in A$ is normal and $B \approx C_0(\Phi_B)$ is the C*-subalgebra it generates,

$$\mathrm{ran}(\widehat{b}) = \sigma_B(b) \cup \{0\} = \sigma_A(b) \cup \{0\},$$

i.e., $\widehat{b}$ maps Φ_B onto the spectrum b of A plus 0. Thus, $f(b)$ only depends on the values of f on $\sigma(b)$.

In particular, if $b \in A$ is normal and $\sigma(b) \subseteq \mathbb{R}_+$, then $\sqrt{b}$ is defined. Moreover, $\sqrt{b} = \sqrt{b}^*$ and $b = \sqrt{b}\sqrt{b} = \sqrt{b}^*\sqrt{b}^*$, so

$$\sigma(b) \subseteq \mathbb{R}_+ \quad \Longrightarrow \quad b \in A_+ = \{a^*a : a \in A\}.$$

The goal is to show that the converse also holds

Proposition 5.36. *For normal $b \in A$,*

$$\sigma(b) \subseteq \mathbb{R}_+ \iff \|\lambda\mathbf{1} - b\| \leq \lambda,$$

for some/all $\lambda \geq \|b\|$.

Proof. Let $B \approx C(\Phi_B \backslash \{0\})$ be the C*-subalgebra of A' generated by b and 1. The ran $\widehat{\lambda\mathbf{1}} = \{\lambda\}$ and ran $\widehat{b} = \sigma(b)$ and hence

$$\|\lambda\mathbf{1} - b\| = \left\|\widehat{\lambda\mathbf{1}} - \widehat{b}\right\| = \lambda - \min(\sigma(b)).$$

Thus, $\|\lambda\mathbf{1} - b\| \leq \lambda$ if and only if $\min(\sigma(b)) \geq 0$ if and only if $\sigma(b) \subseteq \mathbb{B}_+$. $\square$

Corollary 5.37. *For normal $a, b \in A$, if $\sigma(a), \sigma(b) \subseteq \mathbb{R}_+$, then $\sigma(a+b) \subseteq \mathbb{R}_+$.*

Proof. Follows from the previous result since $\|a+b\| \leq \|a\| + \|b\|$ and

$$\|(\|a\| + \|b\|)\mathbf{1} - (a+b)\| \leq \|\|a\|\mathbf{1} - a\| + \|\|b\|\mathbf{1} - b\| \leq \|a\| + \|b\|.$$

□

Theorem 5.38. *For any C*-algebra A, $A_+ = \{b \in A : b \text{ is normal and } \sigma(b) \subseteq \mathbb{R}_+\}$.*

Proof. We already saw that $b \in A_+$ if b is normal and $\sigma(b) \subseteq \mathbb{R}_+$. Conversely, say $b \in A_+$, i.e., $b = a^*a$, for some $a \in A$. Then

$$b^* = (a^*a)^* = a^*a^{**} = a^*a = b$$

so b is normal and $\sigma(b) \subseteq \mathbb{R}$.

Let $b_+ = f_+(b)$ and $b_- = f_-(b)$, where $f_+, f_- \in \mathbb{R}_+^{\mathbb{R}}$ are definded by

$$f_+(r) = 0 \vee r \quad \text{and} \quad f_-(r) = 0 \vee -r.$$

Now,

$$(a\sqrt{b_-})^*a\sqrt{b_-} = \sqrt{b_-}a^*a\sqrt{b_-} = \sqrt{b_-}(b_+ - b_-)\sqrt{b_-} = -b_-^2.$$

Let $a\sqrt{b_-} = x + iy$ for self-adjoint $x, y \in A$. Then

$$a\sqrt{b_-}(a\sqrt{b_-})^* = 2(x^2 + y^2) - (a\sqrt{b_-})^*a\sqrt{b_-}.$$

So by the previous result on sums, $\sigma(a\sqrt{b_-}(a\sqrt{b_-})^*) \subseteq \mathbb{R}_+$ and thus

$$\sigma(-b_-^2) = \sigma((a\sqrt{b_-})^*a\sqrt{b_-}) \subseteq \mathbb{R}_+ \cap \mathbb{R}_- = \{0\}.$$

Thus, $b_- = 0$, i.e., $b = b_+$ and hence $\sigma(b) = \sigma(b_+) \subseteq \mathbb{R}_+$. □

For a C*-algebra A, we have the following

$$\begin{aligned} A_+ &= \{b^*b : b \in A\} \\ &= \{b^2 : b = b^*\} \\ &= \{b : b^*b = bb^* \text{ and } \sigma(b) \subseteq \mathbb{R}_+\}. \end{aligned}$$

In particular, A_+ form a subsemigroup of A, i.e., $A_+ + A_+ \subseteq A_+$.

5.2 GNS-construction

The next goal is to represent non-commutative C*-algebras on Hilber spaces, i.e., as C*-subalgebras of bounded linear operators $\mathcal{B}(H)$.

We use A_+ to define a relation $\leq$ on $A_{\mathrm{sa}} = \{b \in A : b = b^*\}$ by

$$a \leq b \quad \Longleftrightarrow \quad b - a \in A_+.$$

Equivalently, $a \leq b$ means $b \in A_+ + a$. Since A_+ is a (additive) subsemigroup of the (additive) semigroup A_{sa}, $\leq$ is transitive. Since $0 \in A_+$, $\leq$ is also reflexive. Since $A_+ \cap -A_+ = \{0\}$, $\leq$ is also symmetric. Thus $\leq$ is a partial order on A_{sa}.

Proposition 5.39. *For any $a, b, c \in A$, if $a \le b$, then $c^*ac \le c^*bc$.*

Proof. If $a \le b$, then $b - a = d^*d$, for some $d \in A$. Then $c^*bc - c^*ac = c^*(b-a)c = c^*d^*dc = (cd)^*cd \in A_+$. $\square$

Proposition 5.40. *For any $a, b \in A$,*

$$a^*b + b^*a \le a^*a + b^*b,$$
$$(a+b)^*(a+b) \le 2(a^*a + b^*b).$$

Proof. For the first inequality, $0 \le (a-b)^*(a-b) = a^*a - a^*b - b^*a + b^*b$. For the second, $(a+b)^*(a+b) \le a^*a + a^*b + b^*a + b^*b \le 2(a^*a + b^*b)$. $\square$

Definition 5.41. We call $C \subseteq A_+$ a *cone* if $0 \in C = C + C = \mathbb{R}_+ C = C^{\ge}$. We call $I \subseteq A$ a *left ideal* if $0 \in I = I + I = \mathbb{C}I = AI$.

Proposition 5.42. *If $C \subseteq A_+$ is a cone, then $I = \{b \in A : b^*b \in C\}$ is a left ideal. Moreover I is closed if C is closed, since $b \mapsto b^*b$ is continuous.*

Proof. If $a, b \in I$, then $a + b \in I$ since

$$(a+b)^*(a+b) \le 2(a^*a + b^*b) \in \mathbb{R}_+(C + C) \subseteq C.$$

If $a \in I$ and $\lambda \in \mathbb{C}$, then $\lambda a \in I$ since $(\lambda a)^*(\lambda a) = \overline{\lambda}\lambda a^*a \in \mathbb{R}_+ C \subseteq C$. If $a \in A$ and $b \in I$, then $(ab)^*ab = b^*a^*ab \le \|a^*a\| b^*b \in \mathbb{R}_+ C \subseteq C$. Also $0^*0 = 0 \in C$ so $0 \in I$ and hence I is a left ideal. $\square$

Definition 5.43. We call $\phi \in \mathbb{C}^A$ *positive* if $\phi(A_+) \subseteq \mathbb{R}_+$.

Example 5.44. Any character $\psi \in \Phi_A$ is positive.

Proposition 5.45. *Every positive linear ϕ is automatically continuous.*

Proof. Recall that continuous is equivalent to bounded. For all $a \in A$, $a = b + ci$, for $b, c \in A_{\mathrm{sa}}$ with $\|b\|, \|c\| \le \|a\|$. For all $a \in A_{\mathrm{sa}}$, $a = a_+ - a_-$, for $a_+, a_- \in A_+$ with $\|a_+\|, \|a_-\| \le \|a\|$. Thus it suffices for ϕ to be bounded on A_+. If not, we would have $(a_n) \subseteq A_+$ with $\|a_n\| \le \frac{1}{2^n}$ and $\phi(a_n) = 1$. Then $a = \sum a_n \in A_+$ and $\phi(a) \ge \sum \phi(a_n) \to \infty$, a contradiction. $\square$

Any positive linear ϕ defines a positive sesquilinear form $\phi(b^*a)$. Thus the Cauchy-Schwarz inequality gives

$$|\phi(b^*a)|^2 \le \phi(b^*b)\phi(a^*a).$$

Let

$$I = \{b \in A : \phi(b^*b) = 0\} = \{b \in A : \forall a \in A \ \phi(ab) = 0\}.$$

Then I is a left ideal. Cosider the quotient vector space A/I. Let $v_a = a + I$ so $A/I = \{v_a : a \in A\}$. Define $\langle \cdot, \cdot \rangle$ on A/I by

$$\langle v_a, v_b \rangle = \phi(b^*a).$$

This is a well-defined inner product on A/I. Thus, we can complete A/I to obtain a Hilbert space H. Every $a \in A$ determines a map $\pi(a)$ on A/I by

$$\pi(a)(v_b) = v_{ab}.$$

This is well-defined because I is a left ideal. We immediately see that $\pi(a)$ is linear. Also,

$$\|\pi(a)(v_b)\|^2 = \langle v_{ab}, v_{ab}\rangle = \phi(b^*a^*ab) \leq \|a\|^2\phi(b^*b) = \|a\|^2\|v_b\|^2.$$

Thus $\pi(a)$ is bounded and extends to a bounded linear map on H. Also

$$\langle \pi(a)(v_b), v_c\rangle = \langle v_{ab}, v_c\rangle = \phi(c^*ab) = \langle v_b, v_{a^*c}\rangle = \langle v_b, \pi(a^*)(v_c)\rangle.$$

This shows that $\pi(a^*) = \pi(a)^*$, which is its adjoint operator on H. So π represents A on H, i.e., we can identify A with $\pi(A) \subseteq \mathcal{B}(H)$.

So, from positive linear ϕ, we defined a representation $\pi\colon A \to \mathcal{B}(H)$. It remains to show that such positive linear ϕ exists. In the commutative case, we have lots of characters. Indeed, by Gelfand, commutative C*-algebras have the form $C_0(X)$. Then every $x \in X$ gives a character $f \mapsto f(x)$ on $C_0(X)$. Thus, even for non-commutative A, we have lots of characters on the commutative C*-subalgebra generated by any normal $a \in A$. The issue is that these may not extend to characters on A itself. However, they will extend to positive linear ϕ on A. Proving this relies on Hahn-Banach extension theorem.

Theorem 5.46 (Hahn-Banach)**.** *If B is a subspace of a real normed space A, then any $\mathbb{R}$-valued linear ϕ on B with $\|\phi\| = 1$ has a linear extension ψ on A with $\|\psi\| = 1$.*

Proof. By Kuratowski-Zorn, we have a maximal linear extension ψ with $\|\psi\| = \|\phi\|$ on a subspace C containing B. We claim that $C = A$.

If not, take $a \in A \setminus C$ and note that, for any $b, c \in C$,

$$\psi(b) - \psi(c) = \psi(b - c) \leq \|b - c\| = \|b - a + a - c\| \leq \|b - a\| + \|a - c\|.$$

Take $\lambda \in \mathbb{R}$ with

$$\sup_{b \in C}(\psi(b) - \|b - a\|) \leq \lambda \leq \inf_{c \in C}(\|a - c\| + \psi(c)).$$

Define linear θ on $\mathbb{R}a + C \supsetneq C$ by $\theta(\beta a + c) = \beta\lambda + \psi(c)$. Then $\|\theta\| = 1$, contradicting maximality. $\square$

Any $\mathbb{C}$-linear $\psi \in \mathbb{C}^A$ gives $\mathbb{R}$-linear $\phi \in \mathbb{R}^A$ by $\phi(v) = \Re(\psi(v))$. Note that $\|\phi\| = \|\psi\|$:

- Certainly, $\|\phi\| \leq \|\psi\|$ since $|\Re(\lambda)| \leq |\lambda|$, for all $\lambda \in \mathbb{C}$.
- Conversely,

$$1 = \psi(\psi(v)^{-1}v) = \phi(\psi(v)^{-1}v) \leq \|\phi\||\psi(v)^{-1}v| = \|\phi\||\psi(v)|^{-1}|v|.$$

  Thus $|\psi(v)| \leq \|\phi\||v|$, for all v with $\psi(v) \neq 0$, so $\|\psi\| \leq \|\phi\|$.

Also $\psi(v) = \phi(v) - i\phi(iv)$ since $\lambda = \Re(\lambda) - i\Re(i\lambda)$, for all $\lambda \in \mathbb{C}$. Conversely, given any $\mathbb{R}$-linear $\phi \in \mathbb{R}^A$, we can define

$$\psi(v) = \phi(v) - i\phi(iv).$$

Such ψ is $\mathbb{C}$-linear and $\phi(v) = \Re(\phi(v))$. In other word,s we have a norm-preserving correspondence between real linear functionals on A and complex linear functionals on A.

Corollary 5.47. *Any complex linear functional ϕ on a subspace B of a complex normed space A has a linear extension with the same norm on the entirety of A.*

Corollary 5.48. *A linear functional ϕ on a unital C*-algebra A is positive if and only if $\phi(\mathbf{1}) = \|\phi\|$.*

Proof. ($\Longleftarrow$) $\phi(\mathbf{1}) = \|\phi\|$. We first show that ϕ is self-adjoint, i.e., $\phi(A_{\mathrm{sa}}) \subseteq \mathbb{R}$. To see this, take $a \in A_{\mathrm{sa}}$, $n \in \mathbb{Z}$ and let $b = n\mathbf{1} - ia$. Then

$$\|b\|^2 = \|b^*b\| = \|(n\mathbf{1} + ia)(n\mathbf{1} - ia)\| = \left\|n^2\mathbf{1} + a^2\right\| \leq n^2 + \|a\|^2.$$

Let $\phi(a) = r + is$. We need to show that $s = 0$. To see this note

$$\begin{aligned} |\phi(b)|^2 &= |n\phi(\mathbf{1}) - i\phi(a)|^2 = |n\|\phi\| - ir + s|^2 = (n\|\phi\| + s)^2 + r^2 \\ &= n^2\|\phi\|^2 + 2ns\|\phi\| + s^2 + r^2 \leq \|\phi\|^2\|b\|^2 \leq \|\phi\|^2(n^2 + \|a\|^2). \end{aligned}$$

Thus, $2ns\|\phi\| + s^2 + r^2 \leq \|\phi\|^2\|a\|^2$, for all $n \in \mathbb{Z}$, and hence $s = 0$.

To see that ϕ is positive, take $a \in A_+$ with $\|a\| = 1$. By Gelfand, we know that $\|\mathbf{1} - a\| \leq 1$. Thus,

$$\|\phi\| - \phi(a) = \phi(\mathbf{1}) - \phi(a) = \phi(\mathbf{1} - a) \leq \|\phi\|,$$

so $\phi(a) \geq 0$. So if $\phi(\mathbf{1}) = \|\phi\|$, then ϕ is indeed positive.

($\Longrightarrow$) Conversely, assume ϕ is positive and take any $a \in A$ with $\|a\| = 1$. Then $\|a^*a\| = \|a\|^2 = 1$, so $a^*a \leq \mathbf{1}$ by Gelfand. As ϕ is positive, $\phi(a^*a) \leq \phi(\mathbf{1})$ so Cauchy-Schwartz gives

$$|\phi(a)|^2 = |\phi(\mathbf{1}a)|^2 \leq \phi(\mathbf{1})\phi(a^*a) \leq \phi(\mathbf{1})^2.$$

Since a was arbitrary, this shows that $\|\phi\| = \phi(\mathbf{1})$. □

Definition 5.49. A *state* on a C*-algebra is a positive linear functional ϕ with $\|\phi\| = 1$.

Theorem 5.50. *For any $a \in A_+$ we have a state ϕ on A with $\phi(a) = \|a\|$.*

Proof. Consider the C*-subalgebra generated by a and $\mathbf{1}$ in the unitisation $\widetilde{A}$. By Gelfand, this is isomorphic to $C(X)$, for some compact X. Evaluating at some $x \in X$ gives $\|a\|$. So we have a character ϕ on this C*-subalgebra with $\phi(a) = \|a\|$. Note $\phi(\mathbf{1}) = 1 = \|\phi\|$. By Hahn-Banach, ϕ extends to $\widetilde{A}$ without changing the norm. By the previous result, ϕ is therefore positive and hence a state. □

Corollary 5.51. *For any $a \in A$, we have a representation $\pi\colon A \to \mathcal{B}(H)$ with $\|\pi(a)\| = \|a\|$.*

Proof. We may assume A is unital (otherwise unitize then restrict π to A). Take a state ϕ with $\phi(a^*a) = \|a^*a\| = \|a\|^2$. Let π be the GNS representation coming from ϕ. Then

$$\|\pi(a)\|^2 \geq \langle \pi(a)v_{\mathbf{1}}, \pi(a)v_{\mathbf{1}} \rangle = \langle v_a, v_a \rangle = \phi(a^*a) = \|a\|^2.$$

□

To obtain a faithful representation we combine all these π_a, for $a \in A$, i.e., define

$$\pi(b) = \prod_{a\in A} \pi_a(b) \in \prod_{a\in A} \mathcal{B}(H_a) \approx \mathcal{B}(H),$$

where

$$H = \bigoplus_{a\in A} H_a = \left\{(w_a) \in \prod_{a\in A} H_a : \sum_{a\in A} \|w_a\|^2 < \infty\right\}.$$

Then $\pi(a) = \pi(b)$ if and only if $a = b$, i.e., we can identify A with $\pi(A) \subseteq \mathcal{B}(H)$.